# Analysis on pathological spaces: An introduction

lakovos Androulidakis



#### National and Kapodistrian University of Athens

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### Examples

M: compact manifold.

1 Orbits of (some) Lie group actions on M. Vector fields: image of infinitesimal action  $\mathfrak{g} \to \mathfrak{X}(M)$ .

Focus on  $\mathcal{F} = \langle X \rangle$ :

- 2~X nowhere vanishing vector field of  $M \rightsquigarrow$  action of  $\mathbb R$  on M.
- 3 Irrational rotation on torus T<sup>2</sup>: "Kronecker" flow of  $X = \frac{d}{dx} + \theta \frac{d}{dy}$ .  $\mathbb{R}$  injected as a dense orbit (leaf).
- 4 "Horocyclic" foliation:
  - Let  $\Gamma$  cocompact subgroup of  $SL(2, \mathbb{R})$ . Put  $M = SL(2, \mathbb{R})/\Gamma$ .
  - $\mathbb{R}$  is embedded in SL(2,  $\mathbb{R}$ ) by  $\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$ ,  $t \in \mathbb{R}$ .
  - ▶ Therefore  $\mathbb{R}$  acts on M. Action is ergodic,  $\exists$  dense orbits (leaves).

### Laplacians of Kronecker foliation

Kronecker foliation on  $M = T^2$ :  $\mathcal{F} = \langle \frac{d}{dx} + \theta \frac{d}{dy} \rangle$ .  $L = \mathbb{R}$ Two Laplacians:

•  $\Delta_{L} = -\frac{d^{2}}{dx^{2}}$  acting on  $L^{2}(\mathbb{R})$ •  $\Delta_{M} = -X^{2}$  acting on  $L^{2}(M)$ 

By Fourier:

- $\Delta_L \rightsquigarrow$  mult. by  $\xi^2$  on  $L^2(\mathbb{R})$ . Spectrum:  $[0, +\infty)$ .
- $\Delta_M \rightsquigarrow \text{mult. by } (n + \theta k)^2 \text{ on } L^2(\mathbb{Z}^2).$  Spectrum dense in  $[0, +\infty)$ .

Qn 1: Do  $\Delta_L$  and  $\Delta_M$  have the same spectrum for every (regular) foliation?

Qn 2: If so, how to calculate this spectrum?

Tools: Holonomy groupoid  $H(\mathfrak{F})$ , Longitudinal pseudodifferential calculus, Groupoid C\*-algebra(s).

### The C\*-algebra of a Lie groupoid (Connes, Renault) For f, $g \in C_c^{\infty}(G)$ :

- we put  $f^*(x) = \overline{f(x^{-1})}$
- we want to form f \* g by a formula

$$f * g(x) = \int_{yz=x} f(y)g(z)$$

In other words, we want to have an integration along the fibers of the composition  $G \times_{s,t} G \to G$ . Use either Haar systems or half densities.

#### Proposition

The above involution and product make  $C_c^{\infty}(G)$  a \*-algebra.

"Reduced"  $C_r^*(G)$ : completion with left regular representation "Full"  $C^*(G)$ : completion with all representations Quotient  $C^*(G) \rightarrow C_r^*(G)$ .

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# Basic tool: Pseudodifferential calculus (Connes)

The Lie algebra of vector fields tangent to the foliation acts by unbounded multipliers on  $C_c^{\infty}(G)$ . The algebra generated is the algebra of differential operators.

Using Fourier transform one can write a differential operator P (acting by left multiplication on  $f\in C^\infty_c(G))$  as:

$$(\mathsf{Pf})(x, y) = \int \exp(i\langle \varphi(x, z), \xi \rangle) \alpha(x, \xi) \chi(x, z) f(z, y) d\xi dz$$

#### Proposition (Connes)

- ▶ Negative order pseudodifferential operators  $\in C^*(M, F)$
- Zero order pseudodifferential operators: multipliers of C\*(M, F).

Together with multiplicativity of the principal symbol this gives an exact sequence of  $C^*$ -algebras:

$$0 \to C^*(M, F) \to \Psi^0(M, F) \to C(SF^*) \to 0$$

## Laplacians revisited

#### Theorem (Connes, Kordyukov, Vassout)

Elliptic operators of positive order are regular unbounded multipliers (in the sense of Baaj-Woronowicz:  $graph(D) \oplus graph(D)^{\perp}$  is dense).

More generally M compact, (M, F) regular foliation.

- Lie algebra  $\mathfrak{F} = C^{\infty}(M, F)$  acts on  $C^{\infty}(G)$  by unbounded multipliers.
- Laplacian  $\Delta = \sum X_i^2$  is an unbounded (regular) multiplier of  $C^*(M, \mathcal{F})$ .

 $L^2(L), L^2(M)$  are representations of  $C^*(M, \mathfrak{F}).$ 

Proposition (Baaj, Woronowicz)

Every representation extends to regular multipliers.

We recover Laplacians  $\Delta_L$ ,  $\Delta_M$ .

## Statement of 2+1 theorems

Theorem 1 (Connes, Kordyukov)

 $\Delta_M$  and  $\Delta_L$  are essentially self-adjoint.

Also true (and more interesting)

- for  $\Delta_M + f$ ,  $\Delta_L + f$  where f is a smooth function on M. (Schroedinger operators, etc.)
- more generally for every leafwise elliptic (pseudo-)differential operator.

Not trivial because:

- $\Delta_M$  not elliptic (as an operator on M).
- L not compact.

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If L is dense + amenability, \Delta_M and \Delta_L have the same spectrum.
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#### Connes...

In many cases, one can predict the possible gaps in the spectrum.

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# Proof of theorems 1 and 2

#### Theorem 1

 $\Delta_M$  and  $\Delta_L$  are essentially self-adjoint.

- $\blacktriangleright\ L^2(M)$  and  $L^2(L):$  representations of the foliation  $C^*\mbox{-algebras}.$
- Recall (Baaj, Woronowicz): Every representation extends to regular multipliers.

image of the adjoint = adjoint of the image

### Theorem 2 (Kordyukov)

If all leaves L are dense + amenability assumptions,  $\Delta_M$  and  $\Delta_L$  have the same spectrum.

- (Fack and Skandalis): If the foliation is minimal (*i.e.* all leaves are dense) then the foliation C\*-algebra is simple. Whence all representations are faithful.
- Every injective morphism of C\*-algebras is isometric and isospectral.

### Elliptic operators - Gaps of their spectrum

#### Theorem 3 (Connes)

In many cases, one can predict the possible gaps in the spectrum.

More precisely:

- Gaps in the spectrum  $\longrightarrow$  projections in  $C^*(M, F)$ .
- Projectionless  $C^*(M, F)$ : spectrum connected.
- Sometimes dimension function on projections (related with K-theory).
  - Values in  $\mathbb{N}$ : few projections.
  - ▶ values in a dense subset of  $\mathbb{R}_+$ : many projections.

### Examples

#### Horocyclic foliation: no gaps in the spectrum

Let the "ax + b" group act on a compact manifold M. e.g.  $M = SL(2, \mathbb{R})/\Gamma$  where  $\Gamma$  discrete co-compact group. Leaves = orbits of the "x + b" group (assume it is minimal).

The spectrum of the Laplacian is an interval  $[m, +\infty)$ 

#### Proof: We show $C^*(M, F)$ projectionless.

- ▶ ∃ measure on M invariant by ax + b (amenable). x + b invariance  $\implies$  trace on C<sup>\*</sup>(M, F) faithful since C<sup>\*</sup>(M, F) simple (Fack-Skandalis).
- $\blacktriangleright$  The " ax " subgroup  $\longrightarrow$  action of  $\mathbb{R}^*_+$  on  $C^*(M,F)$  which scales the trace.
- Image of  $K_0$  countable subgroup of  $\mathbb{R}$ , invariant under  $\mathbb{R}^*_+$  action.

#### **Similarly**, Kronecker flow: Image of the trace $\mathbb{Z} + \theta \mathbb{Z}$

## Conclusions

Theorems 1 and 2 generalize to any singular foliation!

### Definition (Stefan, Sussmann, A-Skandalis)

A (singular) foliation is a finitely generated sub-module  ${\mathfrak F}$  of  $C^\infty(M;TM),$  stable under brackets.

Examples

- **1**  $\mathbb{R}$  foliated by 3 leaves:  $(-\infty, 0), \{0\}, (0, +\infty)$ .
  - $\mathcal{F}$  generated by  $x^n \frac{\partial}{\partial x}$ . Different foliation for every n.
- 2 R<sup>2</sup> foliated by 2 leaves: {0} and R<sup>2</sup>\{0}.
  No obvious best choice. 𝔅 given by the action of a Lie group

 $\operatorname{GL}(2,\mathbb{R})$ ,  $\operatorname{SL}(2,\mathbb{R})$ ,  $\mathbb{C}^*$ 

IA+Skandalis (2006-today): Holonomy groupoid, foliation C\*-algebras, longitudinal pseudodifferential calculus...

Need to know the shape of  $K_0(C^*(\mathfrak{F}))!$  (Baum-Connes conjecture...)

Careful look at action SO(3)  $\subset \mathbb{R}^3$  (I) dim(Lie(SO(3))) = 3, so  $\mathcal{F} = \operatorname{span}_{C^{\infty}(M)}\langle X, Y, Z \rangle$ .

Take any  $(M, \mathfrak{F})$ . At  $x \in M$  put  $\mathfrak{F}_x = \mathfrak{F}/I_x\mathfrak{F}$ . Get exact sequence

$$0 \to \mathfrak{g}_x \to \mathfrak{F}_x \xrightarrow{ev_x} \mathsf{T}_x \mathsf{L}_x \to 0$$

• 
$$L_x$$
 regular  $\Rightarrow \mathfrak{F}_x = T_x L_x$ 

• 
$$L_x$$
 singular  $\Rightarrow$  dim $(\mathcal{F}_x) >$  dim $(L_x)$ .

•  $dim(\mathcal{F}_{x})$  (upper) semicontinuous

For  $(\mathbb{R}^3, \mathcal{F})$  we have:

- $\mathfrak{F}_0 = \mathfrak{g}_x = \text{Lie}(SO(3))$ , so  $\dim(\mathfrak{F}_0) = 3$
- For  $x \neq 0$ ,  $dim(\mathcal{F}_x) = 2$

$$\mathsf{H}(\mathfrak{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\}$$

Careful look at action SO(3)  $\subset \mathbb{R}^3$  (II)

 $\mathsf{H}(\mathfrak{F}) = (S^2 \times S^2 \times \mathbb{R}^+_*) \cup SO(3) \times \{0\} \text{ decomposes } \mathbb{R}^3:$ 

• 
$$\Omega_1 = \{ x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leq 3 \} = \mathbb{R}^3$$

• 
$$\Omega_0 = \{ x \in \mathbb{R}^3 : \dim(\mathcal{F}_x) \leqslant 2 \} = \mathbb{R}^3 \setminus \{ 0 \}$$

Generalize to arbitrary  $(M, \mathcal{F})$ :

- $dim(\mathfrak{F}_x)$  upper semicontinuous  $\Rightarrow \Omega_i = \{x \in M : dim(\mathfrak{F}_x) \leqslant i\}$  open
- Also,  $Y_i = \Omega_i \setminus \Omega_{i-1}$  closed and saturated.

#### Definition

**1** Decomposition sequence of  $(M, \mathcal{F})$ :

$$\Omega_0 \subseteq \Omega_1 \subseteq \ldots \subseteq \Omega_{k-1} \subseteq \Omega_{\mathbf{k}} = M$$

**2** We say that  $(M, \mathcal{F})$  has height **k**. (**k** =  $+\infty$  allowed and possible!)

# Careful look at action SO(3) $\subset \mathbb{R}^3$ (III)

So foliation  $(\mathbb{R}^3, \mathcal{F})$  has height  $\mathbf{k} = 1$ :

$$\Omega_0 = \mathbb{R}_3 \backslash \{0\}, \qquad \Omega_1 = \mathbb{R}^3, \qquad Y_0 = \Omega_0, \qquad Y_1 = \{0\}.$$

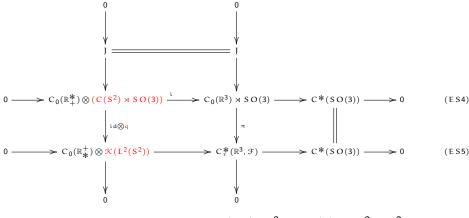
- SO(3) compact, whence amenable:  $C^*(\mathfrak{F}) = C^*_r(\mathfrak{F})$ .
- $C^*(M, \mathcal{F})|_{\Omega_0} = C_0(\mathbb{R}^+_*) \otimes \mathcal{K}(L^2(S^2))$
- $C^*(M, \mathcal{F})|_{Y_1} = C^*(SO(3))$

Exact sequence of C\*-algebras:

$$0 \longrightarrow C_0(\mathbb{R}^+_*) \otimes \mathcal{K}(L^2(S^2)) \longrightarrow C^*(\mathsf{M}, \mathcal{F}) \xrightarrow{\pi_{\mathcal{F}}} C^*(SO(3)) \longrightarrow 0$$

 $SO(3) \subset \mathbb{R}^3$ : calculation (I)

 $\pi: \mathbb{R}^3 \gg \mathrm{SO}(3) \longrightarrow \mathrm{H}(\mathcal{F})$ 



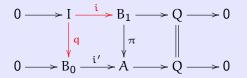
where q: integration along fibers of  $(s, t) : S^2 \rtimes SO(3) \rightarrow S^2 \times S^2$ .

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# Height 1 foliations

#### Proposition

Given a diagram of exact sequences of C\*-algebras and morphisms:



the mapping cone  $\mathcal{C}_{(q,i)}$  of the map  $(q,i) : I \to B_0 \oplus B_1$  is canonically  $E^1$ -equivalent to A (KK-equivalent).

# Height k > 1 foliations

#### Proposition

The previous result extends to foliations  $(M, \mathcal{F})$  of any height: The foliation C\*-algebra is "K"-equivalent (E-equivalent) to a mapping telescope.

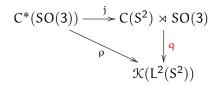
Examples of higher height arise looking at flag manifolds... For instance:

- Let P be the minimal parabolic subgroup of  $GL(n, \mathbb{R})$  (P = uppertriangular matrices).
- Let  $P \times P$  act on  $GL(n, \mathbb{R})$  by left and right multiplication.
- Orbits labeled by symmetric group S<sub>n</sub> (Bruhat decomposition)

# $SO(3) \subset \mathbb{R}^3$ : calculation (II)

 $\rho: C^*(SO(3)) \to \mathcal{K}(L^2(S^2))$  natural repn of SO(3) on  $L^2(S^2).$ 

 $\mathfrak{j}: C^*(SO(3)) \to C(S^2) \rtimes SO(3) \text{ induced by unital inclusion } \mathbb{C} \to C(S^2).$ 



# $SO(3) \subset \mathbb{R}^3$ : calculation (III)

 $C_0(\mathbb{R}^3)=$  mapping cone of  $\mathbb{C}\to C(S^2).$  Taking crossed products by the action of SO(3) and using the first diagram, we find:

+  $C_0(\mathbb{R}^3) \rtimes SO(3)$  in (ES5) is mapping cone  $\mathcal{C}_j$ , where

$$j:C^*(SO(3))\to C(S^2)\rtimes SO(3)$$

 $\blacktriangleright$  Foliation algebra  $C^*(\mathfrak{F})$  in (ES6) is mapping cone  $\mathfrak{C}_\rho.$ 

# $SO(3) \subset \mathbb{R}^3$ : calculation (IV)

So: To calculate  $K_0(C^*(\mathfrak{F}))$ , find kernel of

$$\rho: C^*(SO(3)) \to \mathcal{K}(L^2(S^2)).$$

- $\begin{array}{l} \bullet \mbox{ Peter-Weyl: } C^*(SO(3)) = \oplus_{m \in \mathbb{N}} M_{2m+1}(\mathbb{C}) \mbox{ and } \\ K_0(C^*(SO(3))) = \mathbb{Z}^{(\mathbb{N})} \mbox{ (and } K_1(C^*(SO(3))) = \{0\}). \end{array}$
- ▶ In order to compute the map  $\rho_* : K_0(C^*(SO(3))) \to \mathbb{Z}$ , we have to understand how many times the repn  $\sigma_m (\dim(\sigma_m) = 2m + 1)$  appears in  $\rho$ , *i.e.* count dimension of  $\operatorname{Hom}_{SO(3)}(\sigma_m, \rho)$ .
- Since  $S^2 = SO(3)/S^1$ ,  $\rho = Ind_{S^1}^{SO(3)}(\epsilon)$  where  $\epsilon$  trivial repn of  $S^1$ .
- Frobenius reciprocity thm: dim(Hom<sub>SO(3)</sub>(σ<sub>m</sub>, ρ)) = dim(Hom<sub>S<sup>1</sup></sub>(σ<sub>m</sub>, ε)) = 1.
- So  $\rho_*: K_0(C^*(SO(3))) \to \mathbb{Z}$  maps each generator  $[\sigma_m]$  of  $K_0(C^*(SO(3)))$  to 1.

 $K_0(C^*(\mathfrak{F})) = \ker \rho_* \simeq \mathbb{Z}^{(\mathbb{N})} \qquad K_1(C^*(\mathfrak{F})) = 0$