Perron Frobenius theory and some extensions

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History

Theorem (1)

The dominant eigenvalue of a matrix with positive entries is positive and the corresponding eigenvector could be chosen to be positive.

Theorem (2)

The dominant eigenvalue of an irreducible nonnegative matrix is positive and the corresponding eigenvector could be chosen to be positive.

- O. Perron, *Zur Theorie der Matrizen*, Math. Ann. 64 (1907), 248–263.
- G. Frobenius, *Über Matrizen aus nicht negativen Elementen*, S.-B. Preuss Acad. Wiss. Berlin (1912), 456–477.

Definitions

Definition (1)

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $n \times r$ matrices. Then, $A \geq B \;\; (A > B)$ if $a_{ij} \geq b_{ij} \;\; (a_{ij} > b_{ij})$ for all $i = 1, \ldots, n$, $j = 1, \ldots, r$.

Definition (2)

A matrix $A \in \mathbb{R}^{n,r}$ is said to be *nonnegative (positive) matrix* if $A \geq 0 \;\; (A > 0)$.

Definition (3)

Let $B \in \mathbb{C}^{n,r}$, then $|B|$ denotes the matrix with entries $|b_{ij}|$.

Reducible and Irreducible Matrices

Definition (4)

A matrix $A \in \mathbf{R}^{n,n}$ is said to be *reducible* if there exists a permutation matrix P such that

$$C = PAP^T = \left[\begin{array}{cc} A_{11} & A_{12} \\ 0 & A_{22} \end{array} \right],$$

where $A_{11} \in \mathbf{R}^{r,r}$, $A_{22} \in \mathbf{R}^{n-r,n-r}$ and $A_{12} \in \mathbf{R}^{r,n-r}$, $0 < r < n$.

Definition (5)

A matrix $A \in \mathbf{R}^{n,n}$ is said to be *irreducible* if it is not reducible.

Frobenius normal form

Theorem (3)

For every matrix $A \in \boldsymbol{R}^{n,n}$ *there exists a permutation matrix* $P \in \boldsymbol{R}^{n,n}$ *such that*

$$C = PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix},$$

where each block A_{ii} *is square matrix and is either irreducible or* 1×1 *null matrix.*

The proof is given by considering the 2×2 block form of a reducible matrix. If A_{11} or A_{22} is reducible, we chose the associated permutation matrix to split it again to its 2×2 block form, and so on. Obviously, if A is irreducible then $r = 1$.
The Frobenius normal form is unique, up to a permutation.

Directed Graphs

Definition (6)

The associated *directed graph*, $G(A)$ of an $n \times n$ matrix A, consists of n vertices (nodes) $P_1, P_2, \ldots, P_n$ where an edge leads from P_i to P_j if and only if $a_{ij} \neq 0$.

Definition (7)

A directed graph G is *strongly connected* if for any ordered pair (P_i, P_j) of vertices of G, there exists a sequence of edges (a path), $(P_i, P_{l_1}), (P_{l_1}, P_{l_2}), (P_{l_2}, P_{l_3}), \ldots, (P_{l_{r-1}}, P_j)$ which leads from P_i to P_j. We shall say that such a path has *length* r.

Directed Graphs

Exercise (1)

Let P be a permutation matrix defined by the permutation $(r_1, r_2, \dots, r_n)$ of the integers $(1, 2, \dots, n)$ and $C = PAP^T$ be the permutation transformation of an $n \times n$ matrix A. Prove that the graph $G(C)$ becomes from $G(A)$, by replacing the names of nodes from P_{r_i} to P_i, $i = 1, 2, \dots, n$.

Exercise (2)

Let A be a nonnegative matrix. Prove that the graph $G(A^k)$ consists of all the paths of $G(A)$ of length k.

Theorem (4)

An $n \times n$ matrix A is irreducible iff $G(A)$ is strongly connected.

Proof: Let A is an irreducible matrix. Looking for contradiction, suppose that $G(A)$ is not strongly connected. So there exists an ordered pair of nodes (P_i, P_j) for which there is not connection from P_i to P_j. We denote by S_1 the set of nodes that are connected to P_j and by S_2 the set of remaining nodes. Obviously, there is no connection from any node $P_l \in S_2$ to any node of $P_q \in S_1$, since otherwise $P_l \in S_1$ by definition. Both sets are nonempty since $P_j \in S_1$ and $P_i \in S_2$. Suppose that r and $n - r$ are their cardinalities. Consider a permutation transformation $C = PAP^T$ which reorders the nodes of $G(A)$, such that $P_1, P_2, \ldots, P_r \in S_1$ and $P_{r+1}, P_{r+2}, \ldots, P_n \in S_2$. This means that $c_{kl} = 0$ for all $k = r+1, r+2, \ldots, n$ and $l = 1, 2, \ldots, r$, which constitutes a contradiction since A is irreducible.

Conversely, suppose that A is reducible. Following the above proof in the reverse order we prove that $G(A)$ is not strongly connected.

Theorem (5)

Let $A \geq 0$ be an irreducible $n \times n$ matrix. Then,

- *1. A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.*
- *2. To $\rho(A)$ there corresponds an eigenvector $x > 0$.*
- *3. $\rho(A)$ increases when any entry of A increases.*
- *4. $\rho(A)$ is a simple eigenvalue of A.*
- *5. There is not other nonnegative eigenvector of A different from x.*

First we will prove some other statements.

Lemma (1)

If $A \geq 0$ is an irreducible $n \times n$ matrix, then

$$(I + A)^{n-1} > 0$$

.

Proof: It suffices to prove that $(I + A)^{n-1}x > 0$ for any $x \geq 0$, $x \neq 0$. Define the sequence $x_{k+1} = (I + A)x_k \geq 0$, $k = 0, \ldots, n-2$, $x_0 = x$. Since $x_{k+1} = x_k + Ax_k$, x_{k+1} has no more zero entries than x_k. We will prove that x_{k+1} has fewer zero entries than x_k. Looking for contradiction, suppose that x_{k+1} and x_k have exactly the same zero components. Then, there exists a permutation matrix P such that

$$Px_{k+1} = \begin{pmatrix} y \\ 0 \end{pmatrix}, \quad Px_k = \begin{pmatrix} z \\ 0 \end{pmatrix}, \quad y, z \in \mathbb{R}^m, \quad y, z > 0, \quad 1 \leq m < n.$$

Then,

$$\begin{aligned} Px_{k+1} = \begin{pmatrix} y \\ 0 \end{pmatrix} &= P(x_k + Ax_k) = Px_k + PAP^T Px_k \\ &= \begin{pmatrix} z \\ 0 \end{pmatrix} + \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} z \\ 0 \end{pmatrix}. \end{aligned}$$

This implies that $A_{21} = 0$, which constitutes a contradiction since A is irreducible.
Thus, $x_0 = x$ has at most $n-1$ zero entries, x_k has at most $n-k-1$ zero entries, and consequently,

$$x_{n-1} = (I + A)^{n-1} x_0$$

is a positive vector, which completes the proof.

The Perron Frobenius Theorem

If $A \geq 0$ is an irreducible $n \times n$ matrix, and $x \geq 0$ is any nonzero vector, we define the quantity:

$$r_x = \min_{x_i > 0} \left\{ \frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right\}.$$

Obviously, r_x is a nonnegative real number and is the supremum of all $\rho \geq 0$ for which

$$Ax \geq \rho x.$$

Consider the quantity

$$r = \sup_{x \geq 0, x \neq 0} \{r_x\}. \tag{1}$$

Since $r_x = r_{\alpha x}$ for any $\alpha > 0$, we normalize x such that $\|x\| = 1$ and consider the intersection of the nonnegative hyperoctant and the unit hypersphere: $S = \{x \geq 0 : \|x\| = 1\}$. Then,

$$r = \sup_{x \in S} \{r_x\} = \max_{x \in S} \{r_x\}.$$

Let $Q = \{y : y = (I + A)^{n-1}x, x \in S\}$.
From Lemma (1), Q consists only of positive vectors.
Multiplying the inequality $Ax \geq r_x x$ by $(I + A)^{n-1}$, we obtain $Ay \geq r_x y$, which means that $r_y \geq r_x$.
Thus, the quantity r can be defined equivalently by

$$r = \sup_{y \in Q}\{r_y\} = \max_{y \in Q}\{r_y\}.$$

Since S is a compact set, so is also Q. As r_y is a continuous function on Q, there exists necessarily a positive vector z such that

$$Az \geq rz$$

and no vector w exists for which $Aw > rw$.
We call all such vectors z, *extremal vectors* of the matrix A.

Lemma (2)

If $A \geq 0$ is an irreducible $n \times n$ matrix, the quantity $r = \sup_{x \geq 0, x \neq 0}\{r_x\}$ is positive. Moreover each extremal vector z is a positive eigenvector of the matrix A with corresponding eigenvalue r, i.e., $Az = rz$, $z > 0$.

Proof: For the first part, consider the vector e of all ones. Then,

$$r_e = \min_{e_i > 0}\left\{\frac{\sum_{j=1}^{n} a_{ij} e_j}{e_i}\right\} = \min_{1 \leq i \leq n}\left\{\sum_{j=1}^{n} a_{ij}\right\} > 0 \quad \text{and} \quad r \geq r_e > 0.$$

For the second part, let z be an extremal vector with $Az - rz = \eta \geq 0$. Let $\eta \neq 0$, then

$$(I + A)^{n-1}\eta > 0 \Leftrightarrow Aw - rw > 0, \quad w = (I + A)^{n-1} z > 0.$$

This constitutes a contradiction since it follows that $r_w > r$. Thus $\eta = 0 \Leftrightarrow Az = rz$. Since $w = (I + A)^{n-1} z = (1 + r)^{n-1} z > 0$ it follows that $z > 0$, which completes the proof.

Theorem (6)

Let $A \geq 0$ be an irreducible $n \times n$ matrix, and B be an $n \times n$ complex matrix with $|B| \leq A$. If β is any eigenvalue of B, then

$$|\beta| \leq r, \tag{2}$$

where r is the positive constant of (1). Moreover, equality is valid in (2), i.e., $\beta = re^{\mathrm{i}\phi}$, iff $|B| = A$, and where B has the form

$$B = e^{\mathrm{i}\phi} DAD^{-1}, \tag{3}$$

and D is a diagonal matrix with diagonal entries of modulus unity.

The Perron Frobenius Theorem

Proof: If $\beta y = By, \quad y \neq 0$, then for all $i = 1 \ldots, n$,

$$\beta y_i = \sum_{j=1}^{n} b_{ij} y_j \Rightarrow |\beta||y_i| = |\sum_{j=1}^{n} b_{ij} y_j| \leq \sum_{j=1}^{n} |b_{ij}||y_j| \leq \sum_{j=1}^{n} a_{ij}|y_j|.$$

This means that

$$|\beta||y| \leq |B||y| \leq A|y|, \qquad (4)$$

which implies that $|\beta| \leq r_{|y|} \leq r$, proving (2).
If $|\beta| = r$ then $|y|$ is an extremal vector and consequently it is a positive eigenvector of A corresponding to the eigenvalue r.
Thus,

$$|\beta||y| = |B||y| = A|y|, \qquad (5)$$

and since $|y| > 0$ and $|B| \leq A$, we conclude that

$$|B| = A. \qquad (6)$$

Consider the diagonal matrix

$$D = \left\{ \frac{y_1}{|y_1|}, \frac{y_2}{|y_2|}, \dots, \frac{y_n}{|y_n|} \right\}. \tag{7}$$

Clearly, the diagonal entries of D have modulus unity and $y = D|y|$. Setting $\beta = re^{\mathrm{i}\phi}$,

$$By = \beta y \Leftrightarrow BD|y| = re^{\mathrm{i}\phi} D|y| \Leftrightarrow e^{-\mathrm{i}\phi} D^{-1} BD|y| = r|y|.$$

Let $C = e^{-\mathrm{i}\phi} D^{-1} BD$. Then,

$$C|y| = |B||y| = A|y|. \tag{8}$$

Clearly, $|C| = |B| = A$, hence $C|y| = |C||y|$ and since $|y| > 0$ we obtain that $C = |C| = A$, which proves (3).
Conversely, it is obvious if B is of the form (3), then $|B| = A$, and B has an eigenvalue β with $|\beta| = r$, which completes the proof.

The Perron Frobenius Theorem

Corollary (1)

If $A \geq 0$ is an irreducible $n \times n$ matrix, the positive eigenvalue $r = \sup_{x \geq 0, x \neq 0}\{r_x\}$ is the spectral radius $\rho(A)$ of A.

Proof: Apply Theorem (6) by considering $B = A$.

Corollary (2)

If $A \geq 0$ is an irreducible $n \times n$ matrix and B is any principal square submatrix of A, then $\rho(B) < \rho(A)$.

Proof: Since B is a principal submatrix of A, there exists a permutation matrix P such that $A = \begin{bmatrix} B & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$.

Consider the matrix $C = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$. Clearly, $C \leq A$, $C \neq A$.

Applying Theorem (6) with the matrix C in the place of B we obtain that $\rho(B) = \rho(C) < \rho(A)$.

Proof of the Perron Frobenius Theorem: statements 1, 2, 3

- 1. A has a positive real eigenvalue equal to its spectral radius $\rho(A)$.

Proof: The proof is obtained by Corollary (1).

- 2. To $\rho(A)$ there corresponds an eigenvector $x > 0$.

Proof: The proof is obtained by Lemma (2).

- 3. $\rho(A)$ increases when any entry of A increases.

Proof: Suppose that we increase some entries of A to obtain a new irreducible matrix $\tilde{A}$, where $\tilde{A} \geq A$ and $\tilde{A} \neq A$. Applying Theorem (6) we conclude that $\rho(\tilde{A}) > \rho(A)$.

Proof of the Perron Frobenius Theorem: statement 4

- 4. $\rho(A)$ is a simple eigenvalue of A.

Proof: We use the characteristic polynomial $\Phi(t) = \det(tI - A)$. It is known that

$$\Phi'(t) = \sum_{i=1}^{n} \det(tI - A_i),$$

where A_i is the $(n-1) \times (n-1)$ principal minor of A by deleting the ith row and column. By Corollary (2), $\rho(A_i) < \rho(A)$, which means that $\det(tI - A_i)$ cannot vanish for any $t \geq \rho(A)$. Thus, $\det(\rho(A)I - A_i) > 0$ and consequently

$$\Phi'(\rho(A)) = \sum_{i=1}^{n} \det(\rho(A)I - A_i) > 0.$$

This means that $\rho(A)$ cannot be a zero of $\Phi(t)$ of multiplicity greater than 1, thus $\rho(A)$ is a simple eigenvalue.

Proof of the Perron Frobenius Theorem: statement 5

- 5. There is not other nonnegative eigenvector of A different from x.

Proof: Looking for contradiction, suppose that there exists an eigenvector $y \geq 0$, $y \neq cx$, c constant, corresponding to the eigenvalue λ, $\lambda \neq \rho(A)$.
Multiplying y by $(I + A)^k$, $\quad k \geq n - 1$ we obtain $(I + A)^k y = (1 + \lambda)^k y > 0$, for all $k \geq n - 1$. This means that

$$y > 0 \quad \text{and} \quad \lambda \quad \text{is real with} \quad \rho(A) > \lambda > -1.$$

Since x and y are both positive, we can chose a sufficient small positive t such that $z = x - ty \geq 0$. Then,

$$Az = A(x - ty) = \rho(A)x - \lambda ty > \rho(A)(x - ty) = \rho(A)z.$$

This implies that $r_z > \rho(A)$, which constitutes a contradiction.

Row sums and spectral radius

Theorem (7)

If $A \geq 0$ is an irreducible matrix, then either

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \forall\ i = 1(1)n, \tag{9}$$

or

$$\min_i \left(\sum_{j=1}^{n} a_{ij} \right) < \rho(A) < \max_i \left(\sum_{j=1}^{n} a_{ij} \right). \tag{10}$$

Proof: First suppose that all the row sums are equal. Then, the vector e of all ones is an eigenvector of A: $Ae = \left(\sum_{j=1}^{n} a_{ij} \right) e$. Since $e > 0$, from the Perron Frobenius Theorem (statement 5), it follows that $\rho(A) = \sum_{j=1}^{n} a_{ij}$.

Row sums and spectral radius

If all the row sums are not equal. Then, we construct a nonnegative matrix B by decreasing certain positive entries of A, so that for all $k = 1, 2, \ldots, n$,

$$\sum_{j=1}^{n} b_{kj} = \min_i \left(\sum_{j=1}^{n} a_{ij} \right),$$

where $0 \leq B \leq A$ and $B \neq A$. Then, from the Perron Frobenius Theorem (statement 3), we get $\rho(B) = \min_i \left(\sum_{j=1}^{n} a_{ij} \right) < \rho(A)$. Similarly, we construct an irreducible matrix C by increasing certain positive entries of A, so that for all $k = 1, 2, \ldots, n$,

$$\sum_{j=1}^{n} c_{kj} = \max_i \left(\sum_{j=1}^{n} a_{ij} \right),$$

where $0 \leq A \leq C$ and $C \neq A$. Then, from the Perron Frobenius Theorem (statement 3), we get $\rho(C) = \max_i \left(\sum_{j=1}^{n} a_{ij} \right) > \rho(A)$.

Weighted row sums

Theorem (8)

Let $A \geq 0$ *be an irreducible matrix and* $x \in \boldsymbol{R}^n, x > 0$, *then either*

$$\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} = \rho(A) \;\; \forall \; i = 1(1)n, \tag{11}$$

or

$$\min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) < \rho(A) < \max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right). \tag{12}$$

Moreover,

$$\sup_{x>0} \left\{ \min_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \right\} = \rho(A) = \inf_{x>0} \left\{ \max_i \left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i} \right) \right\}. \tag{13}$$

Weighted row sums

Proof: For any positive vector x, let

$$D = (x_1, x_2, \ldots, x_n).$$

Consider the matrix $B = D^{-1}AD$. B is given by A by similarity transformation, thus B and A have the same eigenvalues. Moreover, $b_{ij} = \frac{a_{ij}xj}{x_i} \geq 0$, thus B is an irreducible nonnegative matrix. Relations (11) and (12) are obtained by applying the previous theorem to the matrix B. From (11) and (12) we have that

$$\sup_{x>0}\left\{\min_i\left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right)\right\} \leq \rho(A) \leq \inf_{x>0}\left\{\max_i\left(\frac{\sum_{j=1}^n a_{ij}x_j}{x_i}\right)\right\}.$$

Equalities are obtained if we chose the positive eigenvector of A, corresponding to $\rho(A)$.

Cyclic and Primitive matrices

Definition (8)

An irreducible nonnegative matrix A is said to be *cyclic* of index $k > 1$, if it has k eigenvalues of modulus equal to $\rho(A)$.

Definition (9)

An irreducible nonnegative matrix A is said to be *primitive*, if the only eigenvalue of A of modulus $\rho(A)$ is $\rho(A)$.

Example (1)

The matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\rho(A) = 1$ and $\lambda = -1 = -\rho(A)$. Thus A is *cyclic* of index 2 or 2-*cyclic*.
The matrix $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues $\rho(B) = \frac{1+\sqrt{5}}{2}$ and $\lambda = \frac{1-\sqrt{5}}{2}$. Obviously, $|\lambda| < \rho(B)$ thus B is *primitive*.

The Perron Frobenius Theorem for Cyclic matrices

Theorem (9)

If an irreducible nonnegative matrix A has k eigenvalues

$$\lambda_0 = re^{i\theta_0}, \lambda_1 = re^{i\theta_1}, \ldots, \lambda_{k-1} = re^{i\theta_{k-1}},$$

of modulus $\rho(A) = r$, $0 = \theta_0 < \theta_1 < \cdots < \theta_{k-1} < 2\pi$, then

- *(a) These numbers are the distinct roots of $\lambda^k - r^k = 0$.*
- *(b) The whole spectrum $S = \{\lambda_0, \lambda_1, \ldots, \lambda_{n-1}\}$ of A goes over into itself under a rotation of the complex plane by $2\pi/k$.*
- *(c) If $k > 1$, then there exists a permutation P such that*

$$PAP^T = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix}. \qquad (14)$$

Proof of (a): Applying Theorem (6) with $B = A$ and $\beta = \lambda_j = re^{i\theta_j}$, the equality condition (3) implies that

$$A = e^{i\theta_j} D_j A D_j^{-1}, \quad |D_j| = I, \quad j = 0, 1, \ldots, k-1. \tag{15}$$

Let $Az = rz, \quad z > 0$ and $y^{(j)} = D_j z$ then

$$Ay^{(j)} = e^{i\theta_j} D_j A D_j^{-1} D_j z = e^{i\theta_j} D_j A z = re^{i\theta_j} D_j z = \lambda_j y^{(j)},$$

so $y^{(j)}$ is an eigenvector corresponding to the simple eigenvalue λ_j. Therefore $y^{(j)}$ and D_j are determined up to a multiplication by scalar.

Define the matrices $D_0, D_1, \ldots, D_{k-1}$ uniquely having their first diagonal element 1. From (15) it follows that

$$\begin{array}{rcl} AD_jD_kz & = & e^{i\theta_j} D_j A D_j^{-1} D_j D_k z = e^{i\theta_j} D_j A y^{(k)} = re^{i(\theta_j+\theta_k)} D_j y^{(k)} \\ & = & re^{i(\theta_j+\theta_k)} D_j D_k z \end{array},$$

$$\begin{array}{rcl} AD_jD_k^{-1}z & = & e^{-i\theta_k} D_k^{-1} A D_k D_j D_k^{-1} z = e^{-i\theta_k} D_k^{-1} A y^{(j)} \\ & = & re^{i(\theta_j-\theta_k)} D_k^{-1} y^{(j)} = re^{i(\theta_j-\theta_k)} D_j D_k^{-1} z. \end{array}$$

Thus, $re^{\mathrm{i}(\theta_j \pm \theta_k)}$ is an eigenvalue of A with eigenvector $D_j D_k^{\pm 1} z$. This means that the numbers $1 = e^{\mathrm{i}\theta_0}, e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_{k-1}}$, and the matrices $I = D_0, D_1, \ldots, D_{k-1}$ are isomorphism abelian multiplicative groups of order k. Thus the numbers $e^{\mathrm{i}\theta_j}$ are k'th roots of unity and $D_j^k = I$.

Proof of (b): It follows from (a) since $e^{\mathrm{i}2\pi/k} A = D_1^{-1} A D_1$ and the fact that the spectrum of a matrix is invariant under a similarity transformation.

Proof of (c): Let $D = D_1$. Since $D^k = I$, the diagonals of D are roots of unity. We permute D such that

$$PDP^T = \begin{bmatrix} e^{\mathrm{i}\delta_0} I_0 & 0 & \cdots & 0 \\ 0 & e^{\mathrm{i}\delta_1} I_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\mathrm{i}\delta_{s-1}} I_{s-1} \end{bmatrix}, \tag{16}$$

where I_j are identity matrices and

$\delta_j = (2\pi/k) n_j, \quad 0 = n_0 < n_1 < \cdots < n_{s-1} < k.$

The Perron Frobenius Theorem for Cyclic matrices

Consider the the same permutation to A:

$$PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} \\ A_{21} & A_{22} & \cdots & A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s1} & A_{s2} & \cdots & A_{ss} \end{bmatrix},$$

where A_{jj} is of the same order as I_{j-1}. From (15) we have $A = e^{\mathrm{i}2\pi/k} DAD^{-1}$, or $PAP^T = e^{\mathrm{i}2\pi/k}(PDP^T)(PAP^T)(PD^{-1}P^T)$ which in matrix form is

$$PAP^T = e^{\mathrm{i}2\pi/k} \begin{bmatrix} A_{11} & e^{\mathrm{i}(\delta_0-\delta_1)}A_{12} & \cdots & e^{\mathrm{i}(\delta_0-\delta_{s-1})}A_{1s} \\ e^{\mathrm{i}(\delta_1-\delta_0)}A_{21} & A_{22} & \cdots & e^{\mathrm{i}(\delta_1-\delta_{s-1})}A_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ e^{\mathrm{i}(\delta_{s-1}-\delta_0)}A_{s1} & e^{\mathrm{i}(\delta_{s-1}-\delta_1)}A_{s2} & \cdots & A_{ss} \end{bmatrix}.$$

Equating the above matrices we obtain a system of s^2 equations

$$A_{pq} = e^{\mathrm{i}2\pi/k} e^{\mathrm{i}(\delta_{p-1}-\delta_{q-1})} A_{pq} \Leftrightarrow A_{pq} = e^{\mathrm{i}2\pi(n_{p-1}-n_{q-1}+1)/k} A_{pq}.$$

Thus, $A_{pq} \neq 0$ iff $n_q = n_p + 1 \mod k$. Since A is irreducible, there exists, for every p, a q such that $A_{pq} \neq 0$ but since $0 = n_0 < n_1 < \cdots < n_{s-1} < k$ it follows that $s = k$ and $n_i = i, \quad i = 0, 1, \ldots, k-1$ and $A_{pq} \neq 0$ only when $q = p + 1 \mod k$. This proves the cyclic form (14) of A.

$$PAP^T = \begin{bmatrix} 0 & A_{12} & 0 & \cdots & 0 \\ 0 & 0 & A_{23} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & A_{k-1,k} \\ A_{k1} & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Eigenvectors of Cyclic matrices

Theorem (10)

Let $A \geq 0$ be an irreducible k-cyclic matrix and z be the eigenvector corresponding to $\rho(A)$, given in the cyclic form:

$$A = \begin{bmatrix} 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & A_{k-1} \\ A_k & 0 & 0 & \cdots & 0 \end{bmatrix}, \quad z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix} > 0.$$

Then the eigenvector $y^{(j)}$ corresponding to $\lambda_j = re^{\mathrm{i}2\pi j/k}$ is

$$y^{(j)} = \begin{bmatrix} z_1 \\ e^{\mathrm{i}2\pi j/k} z_2 \\ e^{\mathrm{i}2\pi 2j/k} z_3 \\ \vdots \\ e^{\mathrm{i}2\pi(k-1)j/k} z_k \end{bmatrix}.$$

Proof: We supposed for simplicity that A is given in its cyclic partition form. If it is not, then permutation transformation is needed to obtain the same result. Consider the matrices D_j defined in Theorem (9), then $y^{(j)} = D_j z$. After the results obtained from Theorem (9), the matrix $D = D_1$ given in (16) is:

$$D = \begin{bmatrix} I_0 & 0 & \cdots & 0 \\ 0 & e^{i2\pi/k} I_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i2\pi(k-1)/k} I_{k-1} \end{bmatrix}.$$

Since the set of D_j's constitutes an isomorphism abelian group, $D_2 = D_1 * D_1 = D^2$, $D_3 = D_2 * D_1 = D^3$, and by induction we obtain that $D_j = D_{j-1} * D_1 = D^j$. Thus,

$$y^{(j)} = \begin{bmatrix} I_0 & 0 & \cdots & 0 \\ 0 & e^{i2\pi j/k} I_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i2\pi(k-1)j/k} I_{k-1} \end{bmatrix} \times \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_k \end{bmatrix} = \begin{bmatrix} z_1 \\ e^{i2\pi j/k} z_2 \\ \vdots \\ e^{i2\pi(k-1)j/k} z_k \end{bmatrix}$$

Characteristic polynomial of Cyclic matrices

Theorem (11)

Let $\lambda^n + a_1\lambda^{n_1} + a_2\lambda^{n_2} \cdots + a_s\lambda^{n_s}$, where $a_1, \ldots, a_s \neq 0$ and $n > n_1 > n_2 > \cdots > n_s$, be the characteristic polynomial of an irreducible matrix $A \geq 0$ which is cyclic of index k. Then, k is the greatest common divisor of the differences $n - n_1, n_1 - n_2, \ldots, n_{s-1} - n_s$.

Proof: Let m be an integer such that A and $e^{i2\pi/m}A$ are similar. Then, for the characteristic polynomials we have

$$\lambda^n + a_1\lambda^{n_1} + \cdots + a_s\lambda^{n_s} = \lambda^n + a_1\theta^{(n-n_1)}\lambda^{n_1} + \cdots + a_s\theta^{(n-n_s)}\lambda^{n_s},$$

where $\theta = e^{i2\pi/m}$. Thus, $a_j = a_j\theta^{(n-n_j)}, \quad j = 1, \ldots, s$ which means that m divides each of the differences $n - n_1, n - n_2, \ldots, n - n_s$. Conversely, if m divides each of the differences, A and $e^{i2\pi/m}A$ have the same spectrum. By Theorem (9) it happens for $m = k$ but not for $m > k$. Thus,

$$k = g.c.d(n-n_1, n-n_2, \ldots, n-n_s) = g.c.d(n-n_1, n_1-n_2, \ldots, n_{s-1}-n_s).$$

Cyclic and Primitive matrices

Corollary (3)

If the trace of an irreducible nonnegative matrix A is positive, then A is primitive.

Corollary (4)

The trace of a cyclic matrix of index $k > 1$, is zero.

Corollary (5)

If $A > 0$, then A is primitive.

Theorem (12)

If a nonnegative irreducible matrix A is primitive, then A^m is also primitive for all positive integers m.

Proof: Since $\rho(A)$ is a simple eigenvalue and the only one with modulus $\rho(A)$, then $(\rho(A))^m$ is a simple eigenvalue of A^m and the only one with modulus $(\rho(A))^m$. Thus, A^m is not a cyclic matrix. It suffices to prove that A^m is irreducible. Let that there exists ν for which A^ν is reducible and is given in its reducible form: $A^\nu = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$. Since $A \geq 0$ is irreducible there exists an eigenvector $x > 0$ corresponding to $\rho(A)$. Then,

$$A^\nu x = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (\rho(A))^\nu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

This gives that $Dx_2 = (\rho(A))^\nu x_2$, thus $(\rho(A))^\nu$ is an eigenvalue of D. Similarly, for A^T there exists $y > 0$ such that

$${A^\nu}^T y = \begin{bmatrix} B^T & 0 \\ C^T & D^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = (\rho(A))^\nu \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Thus $B^T y_1 = (\rho(A))^\nu y_1$, and $(\rho(A))^\nu$ is an eigenvalue of B, which is a contradiction since $(\rho(A))^\nu$ is simple eigenvalue of A^ν.

Lemma (3)

If $A \geq 0$ is an irreducible $n \times n$ matrix with $a_{ii} > 0$, for all $i = 1, 2, \ldots, n$ then $A^{n-1} > 0$.

Proof: The proof follows by Lemma (2). We construct an irreducible nonnegative matrix B, having the same off diagonal structure of A, such that $A \geq \gamma(I + B), \quad \gamma > 0$, obviously $0 < \gamma \leq \min_i a_{ii}$. Then,

$$A^{n-1} \geq \gamma^{n-1}(I + B)^{n-1} > 0,$$

since $(I + B)^{n-1} > 0$ from Lemma (2).

Theorem (13)

Let $A \geq 0$ be an $n \times n$ matrix. Then, $A^m > 0$ for some integer m iff A is primitive.

Proof: Suppose first that $A^m > 0$ for some integer m. Then, A is irreducible since otherwise A^m would be reducible, contradicting to the hypothesis $A^m > 0$. If A is not primitive, then it is cyclic of some index $k > 1$. Thus, there are k eigenvalues of A of modulus $\rho(A)$, and thus there are k eigenvalues of A^m of modulus $\rho(A)^m$. This contradicts Corollary (5) ($A^m > 0 \Rightarrow A^m$ is primitive).

Conversely, if A is primitive then, by definition, it is irreducible. Thus, there exists an identity path

$$(P_1, P_{i_1}), (P_{i_1}, P_{i_2}), \ldots, (P_{i_{r_1}-1}, P_1)$$

of length r_1 in $G(A)$. This means that there exists an identity edge (P_1, P_1) in $G(A^{r_1})$, implying that $(A^{r_1})_{11} > 0$. Similarly, since A^{r_1} is irreducible, there exists an identity path

$$(P_2, P_{j_1}), (P_{j_1}, P_{j_2}), \ldots, (P_{j_{r_2}-1}, P_2)$$

of length r_2 in $G(A^{r_1})$, thus there exists an identity edge (P_2, P_2) in $G(A^{r_1 r_2})$. This implies that $(A^{r_1 r_2})_{11} > 0$ and $(A^{r_1 r_2})_{22} > 0$. Continuing in this way we obtain that $A^{r_1 r_2 \cdots r_n}$ has its diagonal entries positive. Then, Lemma (3) completes the proof.

Definition (10)

The smallest integer m for which $A^m > 0$ is called, *index of primitivity*.

Weakly Cyclic matrices

Definition (11)

An $n \times n$ complex matrix A (not necessarily nonnegative or irreducible) is *weakly cyclic of index* $k > 1$ if there exists a permutation matrix P such that PAP^T is of the cyclic form (14).

Theorem (14)

Let A be an $n \times n$ weakly cyclic matrix of index $k > 1$. Then, A^{jk} is completely reducible for every $j \geq 1$.

$$PA^{jk}P^T = \begin{bmatrix} C_1^j & 0 & \cdots & 0 \\ 0 & C_2^j & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & C_k^j \end{bmatrix}, \tag{17}$$

where $\rho(C_1) = \rho(C_2) = \cdots = \rho(C_k) = \rho^k(A)$. Moreover, $A \geq 0$ is irreducible and cyclic of index k iff each submatrix C_i is primitive.

Proof: The proof of the form (17) is obvious by taking the powers of the matrix A or by using graph theory.
Suppose first that A is irreducible and cyclic of index k.
Looking for contradiction suppose that C_i is cyclic of index l, for some i. Then, since all C_i's have the same nonzero eigenvalues, they should be all cyclic of index l. It means that there exists a permutation matrix P such that $PA^{lk}P^T$ has lk diagonal blocks. This constitutes a contradiction since A would be cyclic of index lk.

Conversely, if C_i is primitive and $A \geq 0$ has the form (14), we have to prove that A is irreducible, since the cyclicity is then obtained from the form (14). Looking for contradiction suppose that A is reducible. Let S_i be the set of m_i nodes corresponding to the ith diagonal block. Then, each node $P_{i_j} \in S_i$ has connection by simple edges only to nodes of S_{i+1}, $i = 1, 2, \ldots, k$, where $S_{k+1} = S_1$. Since A is reducible, there exists an i and a node $P_{i_1} \in S_i$ which has not connection to a node $P_{i_2} \in S_{i+1}$. Suppose that there is a node $P_{i_3} \in S_{i+1}$ such that $(P_{i_1}, P_{i_3}) \in G(A)$, then there is not connection from P_{i_3} to P_{i_2} with a path of length jk, since otherwise there should be a connection from P_{i_1} to P_{i_2} with a path of length $jk+1$. This means that there is not connection from P_{i_3} to P_{i_2} in $G(C_{i+1})$ which contradicts to the hypothesis that C_{i+1} is primitive. If there exists not such an edge $(P_{i_1}, P_{i_3}) \in G(A)$, then P_{i_1} has not connection to any other node. This means that there is a zero row of C_i which leads to the same contradiction.

Theorem (15)

Let $A \geq 0$ be an $n \times n$ irreducible matrix . Let S_i be the set of all the lengths m_i of the identity paths of the node P_i in the directed graph $G(A)$. Let

$$k_i = g.c.d_{m_i \in S_i}\{m_i\}.$$

Then, $k = k_1 = k_2 = \cdots = k_n$ is the index of cyclicity of A. (If $k = 1$ then A is primitive).

Proof: From Theorem (14) it follows that if A is cyclic of index $k > 1$, then only the powers A^{jk} have positive entries in the diagonal blocks C_i^j's and C_i are primitive matrices. Thus there exists sufficient large integer l_0 such that $C_i^l > 0, i = 1, 2, \ldots, n$ for all $l \geq l_0$. This means that A^{lk} have all the diagonal entries positive for all $l \geq l_0$. Hence, $k = g.c.d_{m_i \in S_i}\{m_i\}, i = 1, 2, \ldots, n$. If A is primitive then there exists l_0 such that $A^l > 0$ for all $l \geq l_0$. Thus, $k = g.c.d_{m_i \in S_i}\{m_i\} = 1, i = 1, 2, \ldots, n$.

Exercise (3)

Check and characterize (reducible or irreducible and/or cyclic or primitive) the matrices:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

by using directed graphs. Write them in their reducible or cyclic form. Find all the eigenvalues of the matrices A, B and E, by using the Perron Frobenius theory (without calculating them).

Exercise (4)

Let $A \geq 0$ be a primitive $n \times n$ matrix, which is symmetrically nonnegative, i.e., $a_{ij} > 0$ iff $a_{ji} > 0$. Show that

$$\gamma(A) \leq 2n - 2,$$

where $\gamma(A)$ is the index of primitivity.

Exercise (5)

Let $A \geq 0$ be a primitive $n \times n$ matrix. Prove that

$$\lim_{m \to \infty} \left[tr(A^m)\right]^{1/m} = \rho(A).$$

What is the corresponding result for cyclic matrices of index $k > 1$?

Exercise (6)

Let the $n \times n$ circulant matrix:

$$A = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ \alpha_{n-1} & \alpha_0 & \alpha_1 & \cdots & \alpha_{n-3} & \alpha_{n-2} \\ \alpha_{n-2} & \alpha_{n-1} & \alpha_0 & \cdots & \alpha_{n-4} & \alpha_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_2 & \alpha_3 & \alpha_4 & \cdots & \alpha_0 & \alpha_1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-1} & \alpha_0 \end{bmatrix}.$$

Show that the eigenvalues λ_j of A can be expressed as

$$\lambda_j = \alpha_0 + \alpha_1\phi_j + \alpha_2\phi_j^2 + \cdots + \alpha_{n-1}\phi_j^{n-1}, \quad j = 0, 1, \ldots, n-1,$$

where $\phi_j = e^{i2\pi j/n}$.

Example (2)

Consider the $n \times n$ nonnegative matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

In its directed graph every node P_i has connection only to the neighbour nodes P_{i-1} and P_{i+1}. Thus the identity paths of the node P_i are

$$(P_i, P_{i+1}, P_i),\ (P_i, P_{i-1}, P_i),\ (P_i, P_{i+1}, P_{i+2}P_{i+1}, P_i), \ldots .$$

It is obvious that the lengths of the identity paths of P_i are $S_i = \{2, 4, 6, \ldots\}$, for all $i = 1, 2, \ldots, n$. Thus, by Theorem (15) we obtain that A is a 2 cyclic matrix. Consider now all the paths of length 2 to obtain the graph $G(A^2)$. It constitutes from two disjoint subgraphs, the first one of odd nodes and the second of even. We chose the permutation matrix, by renaming the nodes from $\{P_1, P_3, \ldots, P_{n-1}\}$ to $\{P_1, P_2, \ldots, P_{\frac{n}{2}}\}$ and from $\{P_2, P_4, \ldots, P_n\}$ to $\{P_{\frac{n}{2}+1}, P_{\frac{n}{2}+2}, \ldots, P_n\}$, for even n. For $n = 4$ the cyclic form of A is:

$$PAP^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Reducible matrices

Many of the above results of the Perron Frobenius theory are extended to reducible matrices.

It follows from the continues property, since every nonnegative reducible matrix can be approximated by an irreducible (even primitive) one by replacing certain zero entries by an arbitrary small $\epsilon > 0$.

The Perron Frobenius results of reducible matrices are characterized as *weaker* than those of irreducible matrices.

It follows the main general Perron Frobenius theorem:

Theorem (16)

Let $A \geq 0$ be an $n \times n$ matrix. Then,

- *1. A has a nonnegative real eigenvalue equal to its spectral radius $\rho(A)$. Moreover, this eigenvalue is positive unless A is reducible and its Frobenius form is strictly upper triangular matrix.*
- *2. To $\rho(A)$ there corresponds an eigenvector $x \geq 0$.*
- *3. $\rho(A)$ does not decrease when any entry of A is increased.*

Proof: If A is irreducible then the results follow immediately from Theorem (5). If A is reducible, consider the Frobenius normal form of A:

$$C = PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & A_{22} & \cdots & A_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix}.$$

Reducible matrices

If any diagonal submatrix A_{ii} is irreducible, then it has a positive eigenvalue equal to its spectral radius. Similarly, if A_{ii} is a 1×1 null matrix, its single eigenvalue is zero. Obviously, the eigenvalues of A are the eigenvalues of each A_{ii}. Clearly, A has a nonnegative eigenvalue equal to its spectral radius. If $\rho(A) = 0$, then each A_{ii} is a 1×1 null matrix, thus A is strictly upper triangular.

The other statements follow by applying continuity argument to the results of Theorem (5). We replace some zero entries to $\epsilon > 0$ such that $A(\epsilon)$ becomes irreducible. Then, from Theorem (5), $x(\epsilon) > 0$ is an eigenvector corresponding to $\rho(A(\epsilon))$. Obviously, by taking the limit as ϵ tends to 0 we have

$$\lim_{\epsilon \longrightarrow 0} x(\epsilon) = x \geq 0.$$

For Statement 3, we observe that if an entry of A, belonging to an off diagonal block or to a diagonal block not corresponding to $\rho(A)$, is increased then $\rho(A)$ remains unchanged.

Theorem (17)

Let $A, \in \mathbb{R}^{n,n}$ be a nonnegative matrix and $B \in \mathbb{R}^{n,n}$ be a complex matrix such that $0 \leq |B| \leq A$. Then

$$\rho(B) \leq \rho(A). \tag{18}$$

Proof: If A is irreducible then the result follows from Theorem (6). If A is reducible, we apply the same permutation transformation to A and B such that PAP^T be the Frobenius normal form of A. It is obvious that the inequality $0 \leq |B| \leq A$ is invariant under permutation transformation. Then, apply Theorem (6) to submatrices $|B_{ii}|$ and A_{ii} to obtain the result.

Corollary (6)

If $A \geq 0$ is an $n \times n$ matrix and B is any principal square submatrix of A, then $\rho(B) \leq \rho(A)$.

Proof: As in Corollary (2)

Remark (1)

We observe that the uniqueness property of the spectral radius for irreducible matrices, does not valid in the reducible case.
If there are k diagonal blocks in the Frobenius normal form that have spectral radius $\rho(A)$, then $\rho(A)$ is an eigenvalue of multiplicity k.
To $\rho(A)$, there correspond k eigenvectors or generalized eigenvectors.
We say that the set of these vectors form the eigenspace of $\rho(A)$.
It has been proved that the eigencpace of $\rho(A)$, has a basis of nonnegative vectors.
A question arises here:
When A has a positive eigenvector corresponding to spectral radius?
The following two theorems give an answer to this question.

Theorem (18)

Let $A, \in \boldsymbol{R}^{n,n}$ be a nonnegative reducible matrix and let $S_1, S_2, \ldots, S_r$ be the sets of nodes corresponding to the diagonal blocks $A_{11}, A_{22}, \ldots, A_{rr}$ in the Frobenius normal form of A. To $\rho(A)$ corresponds a positive eigenvector iff every set S_k corresponding to A_{kk}, with $\rho(A_{kk}) = \rho(A)$, has not connection to any other set S_j, while every set S_i corresponding to A_{ii}, with $\rho(A_{ii}) < \rho(A)$, has connection to at least one set S_k corresponding to A_{kk}, with $\rho(A_{kk}) = \rho(A)$.

Theorem (19)

Let $A, \in \boldsymbol{R}^{n,n}$ be a nonnegative reducible matrix. To $\rho(A)$ corresponds a positive eigenvector x of A and a positive eigenvector y of A^T iff the Frobenius normal form of A consists of a block diagonal matrix, where every block has spectral radius $\rho(A)$.

The following examples show the validity of the above theorems.

Example (3)

Consider the nonnegative matrix A in its Frobenius normal form:

$$A = \left[\begin{array}{cc|c|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 2 \end{array}\right].$$

The diagonal blocks are $A_{11} = \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array}\right]$ with $\rho(A_{11}) = 2 = \rho(A)$, $A_{22} = [1]$ with $\rho(A_{22}) = 1 < \rho(A)$ and $A_{33} = [2]$ with $\rho(A_{33}) = 2 = \rho(A)$. Obviously, $S_1 = \{P_1, P_2\}$, $S_2 = \{P_3\}$ and $S_3 = \{P_4\}$. Since S_1 and S_3, corresponding to blocks of $\rho(A)$, have not connection to any other set, while S_2 has connection to S_3, A has a positive eigenvector corresponding to $\rho(A)$.
The eigenvectors of A corresponding to $\rho(A)$ are $x^{(1)} = (1\ 1\ 1\ 1)^T > 0$ and $x^{(2)} = (1\ 1\ 0\ 0)^T \geq 0$ and the validity of Theorem (18) is confirmed.

Example (4)

Consider the nonnegative matrix B in its Frobenius normal form:

$$B = \left[\begin{array}{cc|c|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \\ \hline 0 & 0 & 0 & 1 \end{array}\right],$$

which is obtain from A of Example (3) by interchanging the diagonal blocks A_{22} and A_{33}. It is easily checked that B is the Frobenius normal form of A^T. Since S_2, corresponding to block of $\rho(B)$, has connection to S_3, B has not a positive eigenvector corresponding to $\rho(B)$.
The eigenvectors of B corresponding to $\rho(B)$ are $x^{(1)} = (1\ 1\ 1\ 0)^T \geq 0$ and $x^{(2)} = (1\ 1\ 0\ 0)^T \geq 0$. Thus, the validity of both Theorems (18)and (19) is confirmed.

Example (5)

Consider the nonnegative matrix C in its Frobenius normal form:

$$C = \left[\begin{array}{cc|c|c} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 2 \end{array}\right].$$

Since S_1, corresponding to block of $\rho(C)$, has connection to both sets S_2 and S_3, A has not a positive eigenvector corresponding to $\rho(C)$. We observe also that two sets that correspond to $\rho(C)$ are connected ($S_1 \rightarrow S_3$). For this reason C has not two linearly independent eigenvectors. It has one eigenvector $x^{(1)} = (1\ 1\ 0\ 0)^T \geq 0$ and one generalized eigenvector $x^{(2)} = (1\ 2\ 2\ 2)^T > 0$ and the validity of Theorem (18) is confirmed.

Observe that these vectors constitute a nonnegative basis of the eigencpace of $\rho(C)$.

Example (6)

Consider the nonnegative matrix D in its Frobenius normal form:

$$D = \left[\begin{array}{cc|c|cc} 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & 0 & 0 \\ \hline 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right].$$

D is a block diagonal matrix with all diagonal blocks having spectral radii equal to $\rho(D) = 3$. Thus by Theorem (19), both matrices D and D^T have a positive eigenvector corresponding to $\rho(D)$.
The eigenvectors of D are: $x^{(1)} = (1\ 1\ 1\ 1\ 1)^T > 0$, $x^{(2)} = (1\ 1\ 1\ 0\ 0)^T \geq 0$ and $x^{(3)} = (1\ 1\ 0\ 0\ 0)^T \geq 0$.
The eigenvectors of D^T are: $y^{(1)} = (2\ 1\ 1\ 2\ 1)^T > 0$, $y^{(2)} = (2\ 1\ 1\ 0\ 0)^T \geq 0$ and $y^{(3)} = (2\ 1\ 0\ 0\ 0)^T \geq 0$.

Reducible matrices

Exercise (7)

Construct the Frobenius normal form of the matrix:

$$A = \begin{bmatrix} 1 & 8 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find the spectral radius $\rho(A)$ and its eigenspace.

Theorem (20)

If $A \geq 0$ is an $n \times n$ matrix, then either

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \ \ \forall \ i = 1(1)n, \tag{19}$$

or

$$\min_i \left(\sum_{j=1}^{n} a_{ij} \right) \leq \rho(A) \leq \max_i \left(\sum_{j=1}^{n} a_{ij} \right). \tag{20}$$

Reducible matrices - Weighted row sums and spectral radius

Theorem (21)

Let $A \geq 0$ *be an* $\times n$ *matrix and* $x \in \boldsymbol{R}^n, x > 0$, *then either*

$$\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} = \rho(A) \ \ \forall \ i = 1(1)n, \tag{21}$$

or

$$\min_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right). \tag{22}$$

Theorem (22)

Let $A \in \mathbf{R}^{n,n}$ be a nonnegative matrix and $x \geq 0$ $(x \neq 0)$ be such that $Ax - \alpha x \geq 0$ for a constant $\alpha > 0$. Then

$$\alpha \leq \rho(A). \tag{23}$$

Moreover, if $Ax - \alpha x > 0$, then the inequality in (23) is strict.

Proof: If A is irreducible, then by the definition of the quantity r_x in the proof of the Perron Frobenius Theorem, we have that $r_x \geq \alpha$. Thus, $\rho(A) = r \geq r_x \geq \alpha$.
If A is reducible then we use the continuity argument. Replace by ϵ some zero entries, such that $A(\epsilon)$ becomes irreducible. Then $A(\epsilon)x \geq Ax \geq \alpha x$, Thus $\rho(A(\epsilon)) \geq \alpha$ and by taking the limit we get the result.
If $Ax - \alpha x > 0$ then there exists $\beta > \alpha$ such that $Ax - \beta x \geq 0$. Then the result becomes from the previous proof.

Theorem (23)

Let $A \in \mathbf{R}^{n,n}$ be a nonnegative matrix and $x > 0$ be such that $\alpha x - Ax \geq 0$ for a constant $\alpha > 0$. Then

$$\rho(A) \leq \alpha. \tag{24}$$

Moreover, if $\alpha x - Ax > 0$, then the inequality in (24) becomes strict.

Proof: Let $y \geq 0$ be the eigenvector of A^T corresponding to $\rho(A)$. We premultiply the inequality by y^T to get

$$y^T(\alpha x - Ax) \geq 0 \Leftrightarrow \alpha y^T x - \rho(A) y^T x \geq 0 \Leftrightarrow \alpha - \rho(A) \geq 0,$$

which proves our assertion. The strict inequality is obvious.
We remark that the condition $x > 0$ is necessary, since for $x \geq 0$ and A reducible such that $Ax = 0$, $\alpha x - Ax \geq 0$ holds for any $\alpha \geq 0$, but (24) is not true for all $\alpha \geq 0$.

Extension of the Perron-Frobenius theory

It is obvious, from the continuity, that the Perron-Frobenius theory may hold also in the case where the matrix has some absolutely small negative entries.
This observation brings up some questions:

- How small could these entries be?
- What is their distribution?
- When such a matrix looses the Perron-Frobenius property?

Tarazaga et' al gave a partial answer to the first question by providing a sufficient condition for the symmetric matrix case:

$$e^T A e \geq \sqrt{(n-1)^2 + 1} \|A\|_F, \quad e = (1\ 1 \cdots 1)^T$$

- P. Tarazaga, M Raydan and A. Hurman, *Perron-Frobenius theorem for matrices with some negative entries*, Linear Algebra Appl. 328 (2001), 57–68.

Extension of the Perron-Frobenius theory

Our aim is:

- To answer implicitly the above questions by extending the **Perron-Frobenius theory** of nonnegative matrices to the class of matrices that **possess the Perron-Frobenius property.**
- To Study when and in which applications, this theory can be applied.

D. Noutsos. *On Perron-Frobenius property of matrices having some negative entries.* Linear Algebra Appl. 412 (2006), 132–153.

Definition (12)

A matrix $A \in \boldsymbol{R}^{n,n}$ possesses the Perron-Frobenius property if its dominant eigenvalue λ_1 is positive and the corresponding eigenvector $x^{(1)}$ is nonnegative.

Definition (13)

A matrix $A \in \boldsymbol{R}^{n,n}$ possesses the strong Perron-Frobenius property if its dominant eigenvalue λ_1 is positive, simple $(\lambda_1 > |\lambda_i|, \ i = 2, 3, \cdots, n)$ and the corresponding eigenvector $x^{(1)}$ is positive.

Definition (14)

A matrix $A \in \boldsymbol{R}^{n,n}$ is said to be *eventually positive (eventually nonnegative)* if there exists a positive integer k_0 such that $A^k > 0$ $(A^k \geq 0)$ for all $k \geq k_0$.

Theorem (24)

For a symmetric matrix $A \in \boldsymbol{R}^{n,n}$ the following properties are equivalent:
(i) A possesses the strong Perron-Frobenius property.
(ii) A is an eventually positive matrix.

Proof: $(i \Rightarrow ii)$: $\lambda_1 = \rho(A) > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$, where λ_1 is a simple, with the eigenvector $x^{(1)} \in \boldsymbol{R}^n$ being positive.
Choose the i^{th} column $a^{(i)} \in \boldsymbol{R}^n$ of A.
Expand $a^{(i)}$: $a^{(i)} = \sum_{j=1}^{n} c_j x^{(j)}$.
$c_j = (a^{(i)}, x^{(j)})$, $j = 1, 2, \ldots, n$, So, $c_1 = (a^{(i)}, x^{(1)}) = \lambda_1 x_i^{(1)} > 0$.
Apply Power method: $\lim_{k \to \infty} A^k a^{(i)} > 0 \Rightarrow A^k a^{(i)} > 0 \ \forall k \geq m$.
Choose $m_0 = \max_i \left\{ m \mid A^k a^{(i)} > 0 \ \forall \ k \geq m \right\}$, then, $A^k > 0$ for all $k \geq k_0 = m_0 + 1$. So, A is an eventually positive matrix.

$(ii \Rightarrow i)$: From the Perron-Frobenius theory of nonnegative matrices, the assumption $A^k > 0$ means that the dominant eigenvalue of A^k is positive and the only one in the circle while the corresponding eigenvector is positive. It is well known that the matrix A has as eigenvalues the k^{th} roots of those of A^k with the same eigenvectors. Since this happens $\forall k \geq k_0$, A possesses the strong Perron-Frobenius property.

Example (7)

Let $A = \begin{pmatrix} -1 & 8 & -1 \\ 8 & 8 & 8 \\ -1 & 8 & 8 \end{pmatrix}$.

$$e^T A e = 45, \ \sqrt{(n-1)^2 + 1}\|A\|_F = 43.9886.$$

$$A^k > 0, \quad k \geq 2$$

$$\lambda_1 = 17.5124, \ \lambda_2 = -7.4675, \ \lambda_3 = 4.9551,$$

$$x = (0.2906 \ \ 0.7471 \ \ 0.5978)^T.$$

A **possesses the strong Perron-Frobenius property.**

Example (8)

While if $A = \begin{pmatrix} -4 & 8 & -4 \\ 8 & 8 & 8 \\ -4 & 8 & 8 \end{pmatrix}$.

$$e^T A e = 36, \;\; \sqrt{(n-1)^2+1}\|A\|_F = 46.4758.$$

$$A^k > 0, \quad k \geq 8$$

$$\lambda_1 = 16.4959, \;\; \lambda_2 = -10.9018, \;\; \lambda_3 = 6.4059,$$

$$x = (0.1720 \;\; 0.7563 \;\; 0.6312)^T.$$

A possesses the strong Perron-Frobenius property.

Theorem (25)

For a matrix $A \in \boldsymbol{R}^{n,n}$ the following properties are equivalent:
i) Both matrices A and A^T possess the strong Perron-Frobenius property.
ii) A is an eventually positive matrix.
iii) A^T is an eventually positive matrix.

Proof: $(i \Rightarrow ii)$: Let $A = XDX^{-1}$ be the Jordan canonical form of the matrix A. We assume that the eigenvalue $\lambda_1 = \rho(A)$ is the first diagonal entry of D. So the Jordan canonical form can be written as

$$A = [x^{(1)} | X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array} \right] \left[\begin{array}{c} {y^{(1)}}^T \\ \hline Y_{n-1,n} \end{array} \right], \tag{25}$$

where ${y^{(1)}}^T$ and $Y_{n-1,n}$ are the first row and the matrix formed by the last $n-1$ rows of X^{-1}, respectively. Since A possesses the strong Perron-Frobenius property, the eigenvector $x^{(1)}$ is positive. From (25), the block form of A^T is

$$A^T = [y^{(1)} | Y_{n-1,n}^T] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1}^T \end{array} \right] \left[\begin{array}{c} {x^{(1)}}^T \\ \hline X_{n,n-1}^T \end{array} \right]. \tag{26}$$

The matrix $D_{n-1,n-1}^T$ is the block diagonal matrix formed by the transposes of all Jordan blocks except λ_1. It is obvious that there exists a permutation matrix $P \in \mathbb{R}^{n-1,n-1}$ such that the associated permutation transformation on the matrix $D_{n-1,n-1}^T$ transposes all the Jordan blocks.

Thus, $D_{n-1,n-1} = P^T D^T_{n-1,n-1} P$ and relation (26) takes the form:

$$
\begin{aligned}
A^T &= [y^{(1)}|Y^T_{n-1,n}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array}\right] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D^T_{n-1,n-1} \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P \end{array}\right] \\
&\times \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & P^T \end{array}\right] \left[\begin{array}{c} {x^{(1)}}^T \\ \hline X^T_{n,n-1} \end{array}\right] = [y^{(1)}|{Y'}^T_{n-1,n}] \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & D_{n-1,n-1} \end{array}\right] \left[\begin{array}{c} {x^{(1)}}^T \\ \hline {X'}^T_{n,n-1} \end{array}\right]
\end{aligned}
$$

where ${Y'}^T_{n-1,n} = Y^T_{n-1,n} P$ and ${X'}^T_{n,n-1} = P^T X^T_{n,n-1}$. The last relation is the Jordan canonical form of A^T which means that $y^{(1)}$ is the eigenvector corresponding to the dominant eigenvalue λ_1. Since A^T possesses the strong Perron-Frobenius property, $y^{(1)}$ is a positive vector or a negative one. Since ${y^{(1)}}^T$ is the first row of X^{-1} we have that $(y^{(1)}, x^{(1)}) = 1$ implying that $y^{(1)}$ is a positive vector.

We return now to the Jordan canonical form (25) of A and form the power A^k.

$$A^k = [x^{(1)} | X_{n,n-1}] \left[\begin{array}{c|c} \lambda_1^k & 0 \\ \hline 0 & D_{n-1,n-1}^k \end{array} \right] \left[\begin{array}{c} {y^{(1)}}^T \\ \hline Y_{n-1,n} \end{array} \right]$$

or

$$\frac{1}{\lambda_1^k} A^k = [x^{(1)} | X_{n,n-1}] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \frac{1}{\lambda_1^k} D_{n-1,n-1}^k \end{array} \right] \left[\begin{array}{c} {y^{(1)}}^T \\ \hline Y_{n-1,n} \end{array} \right].$$

Since λ_1 is the dominant eigenvalue, the only one of modulus λ_1, we get that $\lim_{k\to\infty} \frac{1}{\lambda_1^k} D_{n-1,n-1}^k = 0$. Thus,

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} {y^{(1)}}^T > 0.$$

The last relation means that there exists an integer $k_0 > 0$ such that $A^k > 0$ for all $k \geq k_0$. So, A is an eventually positive matrix and the first part of Theorem is proved.
$(ii \Leftrightarrow iii)$: Obvious from Definition (14)
$(ii \Rightarrow i)$: The proof is the same as that of Theorem (24), by considering that A and A^T are both eventually positive matrices.

Theorem (26)

Let that $A \in \mathbb{R}^{n,n}$ is an eventually nonnegative matrix. Then, both matrices A and A^T possess the Perron-Frobenius property.

Proof: Analogous to the proof of the part $(ii \Rightarrow i)$ of Theorem (24).

Theorem (27)

Let that both the matrices $A \in \mathbb{R}^{n,n}$ and A^T possess the Perron-Frobenius property, with the dominant eigenvalue $\lambda_1 = \rho(A)$ being the only one in the circle $|\lambda_1|$ $(\lambda_1 > |\lambda_i|,\ i = 2, 3, \cdots, n)$. Then,

$$\lim_{k\to\infty} \frac{1}{\lambda_1^k} A^k = x^{(1)} {y^{(1)}}^T \geq 0. \tag{27}$$

Proof: Analogous to the proof of part $(i \Rightarrow ii)$ of Theorem (25).

In concluding, it is noted here that the class of the eventually positive matrices is a subclass of the class of matrices possessing the strong Perron-Frobenius property, while the class of the eventually nonnegative matrices is a subclass of the class of matrices possessing the Perron-Frobenius property. This is shown by the following example.

Example (9)

The matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ -.4 & 1 & 1 \\ -.4 & 5 & 8 \end{pmatrix}$ has dominant eigenvalue 8.5523 with eigenvector $(0.1618\ 0.1211\ 0.9794)^T$. The corresponding eigenvector of A^T is $(0.07308\ \ -0.5371\ \ -0.8404)^T$. As one can readily see, A possesses the strong Perron-Frobenius property while A^T does **not**. According to Theorem (25), A is not an eventualy positive matrix. This is easily checked by seeing that the first column vector of A^k, $k \geq 2$ is negative.

Theorem (28)

If $A^T \in \mathbf{R}^{n,n}$ possesses the Perron-Frobenius property, then either

$$\sum_{j=1}^{n} a_{ij} = \rho(A) \quad \forall\ i = 1(1)n, \tag{28}$$

or

$$\min_i \left(\sum_{j=1}^{n} a_{ij} \right) \le \rho(A) \le \max_i \left(\sum_{j=1}^{n} a_{ij} \right). \tag{29}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (29) are strict.

Row sums and spectral radius

Proof: Let $(\rho(A), y)$ be the Perron-Frobenius eigenpair of the matrix A^T and e the vector of ones. Then,

$$y^TAe = y^T \begin{pmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{nj} \end{pmatrix} = \sum_{i=1}^n \left(y_i \sum_{j=1}^n a_{ij} \right) \leq \max_i \left(\sum_{j=1}^n a_{ij} \right) \sum_{i=1}^n y_i,$$

$$y^TAe = \sum_{i=1}^n \left(y_i \sum_{j=1}^n a_{ij} \right) \geq \min_i \left(\sum_{j=1}^n a_{ij} \right) \sum_{i=1}^n y_i.$$

On the other hand we get

$$y^TAe = e^TA^Ty = \rho(A)e^Ty = \rho(A) \sum_{i=1}^n y_i.$$

Combining the relations above, we get our result. Obviously, equality holds if the row sums are equal. If A^T possesses the strong Perron-Frobenius property, then $y > 0$ and the inequalities become strict.

Row sums and spectral radius

Note that it is necessary to have $\max_i \left(\sum_{j=1}^{n} a_{ij}\right) > 0$, but it is not necessary to have $\min_i \left(\sum_{j=1}^{n} a_{ij}\right) \geq 0$ as is shown in the following example.

Example (10)

Let

$$A = \begin{pmatrix} 1 & 1 & -3 \\ -4 & 1 & 1 \\ 8 & 5 & 8 \end{pmatrix}.$$

The vector of the row sums of A is $(-1 \ \ -2 \ \ 21)^T$, while A^T possesses the strong Perron-Frobenius property with the Perron-Frobenius eigenpair:
$(6.868\ ,\ (0.4492 \ \ 0.6225 \ \ 0.6408)^T)$.

Corollary (7)

If $A \in \boldsymbol{R}^{n,n}$ possesses the Perron-Frobenius property, then either

$$\sum_{i=1}^{n} a_{ij} = \rho(A) \quad \forall\, j = 1(1)n, \tag{30}$$

or

$$\min_j \left(\sum_{i=1}^{n} a_{ij} \right) \le \rho(A) \le \max_j \left(\sum_{i=1}^{n} a_{ij} \right). \tag{31}$$

Moreover, if A possesses the strong Perron-Frobenius property, then both inequalities in (31) are strict.

Theorem (29)

If $A^T \in \mathbf{R}^{n,n}$ possesses the Perron-Frobenius property and $x \in \mathbf{R}^{n,n}, x > 0$, then either

$$\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} = \rho(A) \;\; \forall\; i = 1(1)n, \tag{32}$$

or

$$\min_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right). \tag{33}$$

Moreover, if A^T possesses the strong Perron-Frobenius property, then both inequalities in (33) are strict and

$$\sup_{x>0} \left\{ \min_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \right\} = \rho(A) = \inf_{x>0} \left\{ \max_i \left(\frac{\sum_{j=1}^{n} a_{ij} x_j}{x_i} \right) \right\}.$$

Two questions come up:

- What happens to the monotonicity in case the matrices possess the Perron-Frobenius property?
- Does the property of "possessing the Perron-Frobenius property" still hold when the entries of A increase, as it does in the nonnegative case?

Theorem (30)

If the matrices $A, B \in \boldsymbol{R}^{n,n}$ are such that $A \leq B$, and both A and B^T possess the Perron-Frobenius property (or both A^T and B possess the Perron-Frobenius property), then

$$\rho(A) \leq \rho(B). \tag{34}$$

Moreover, if the above matrices possess the strong Perron-Frobenius property and $A \neq B$, then (34) becomes strict.

Monotonicity properties

Proof: Let $x, y \geq 0$ be the Perron right and left eigenvectors of A and B associated with the dominant eigenvalues λ_A and λ_B, respectively. Then the following equalities hold

$$y^T Ax = \lambda_A y^T x, \quad y^T Bx = \lambda_B y^T x.$$

Since $A \leq B$, $B = A + C$, where $C \geq 0$. So,

$$\lambda_B y^T x = y^T Bx = y^T (A+C)x = y^T Ax + y^T Cx \geq y^T Ax = \lambda_A y^T x.$$

Assuming that $y^T x > 0$, the above relations imply that $\lambda_B \geq \lambda_A$. The case where $y^T x = 0$ is covered by using a continuity argument and perturbation technique.
It is also obvious that the inequality becomes strict in case the associated Perron-Frobenius properties are strong.

This property does not guarantee the existence of the Perron-Frobenius property for a matrix C ($A \le C \le B$).

Example (11)

$$A = \begin{pmatrix} 1 & 2 & -.2 \\ -.4 & 1 & 1 \\ -.2 & 5 & 8 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 2 & -.1 \\ -.4 & 1 & 1 \\ -.1 & 5 & 8 \end{pmatrix},$$
$$B = \begin{pmatrix} 1 & 2 & .1 \\ -.1 & 1 & 1 \\ .1 & 5 & 8 \end{pmatrix}.$$

A possesses the Perron Frobenius property.
C possesses the Perron Frobenius property.
C^T does **not** possess the Perron Frobenius property.
B^T possesses the Perron Frobenius property.

$$\rho(A) = 8.6499, \quad \rho(C) = 8.6464, \quad \rho(B) = 8.6548.$$

Monotonicity properties

Theorem (31)

Let (**i**) $A^T \in \mathbb{R}^{n,n}$ *possesses the Perron-Frobenius property and* $x \geq 0$ ($x \neq 0$) *be such that* $Ax - \alpha x \geq 0$ *for a constant* $\alpha > 0$ *or* (**ii**) $A \in \mathbb{R}^{n,n}$ *possesses the Perron-Frobenius property and* $x \geq 0$ ($x \neq 0$) *be such that* $x^T A - \alpha x^T \geq 0$ *for a constant* $\alpha > 0$. *Then*

$$\alpha \leq \rho(A). \tag{35}$$

Moreover, if $Ax - \alpha x > 0$ *or* $x^T A - \alpha x^T > 0$, *then the inequality in (35) is strict.*

Proof: (i): Let $y \geq 0$ be the Perron eigenvector of A^T. Then,

$$y^T(Ax - \alpha x) \geq 0 \Longleftrightarrow (\rho(A) - \alpha)y^T x \geq 0.$$

If $y^T x > 0$, then (35) holds. If $y^T x = 0$ we recall the perturbation argument. If $Ax - \alpha x > 0$, (35) becomes strict.
(ii) The proof is similar.

Monotonicity properties

Theorem (32)

Let (**i**) $A^T \in \mathbb{R}^{n,n}$ *possesses the Perron-Frobenius property and* $x > 0$ *be such that* $\alpha x - Ax \geq 0$ *for a constant* $\alpha > 0$ *or* (**ii**) $A \in \mathbb{R}^{n,n}$ *possesses the Perron-Frobenius property and* $x > 0$ *be such that* $\alpha x^T - x^T A \geq 0$ *for a constant* $\alpha > 0$. *Then*

$$\rho(A) \leq \alpha. \tag{36}$$

Moreover, if $\alpha x - Ax > 0$ *or* $\alpha x^T - x^T A > 0$, *then the inequality in (36) becomes strict.*

We remark that the condition $x > 0$ is necessary. This is because for $x \geq 0$ such that $Ax = 0$, the condition $\alpha x - Ax \geq 0$ holds for any $\alpha \geq 0$, but the inequality (36) is not true for all $\alpha \geq 0$.

Monotonicity properties

Theorem (33)

Let $A \in \mathbb{R}^{n,n}$ possesses the Perron-Frobenius property with $x \geq 0$ the associated eigenvector and let $y \neq 0$ such that $y^T x > 0$. Then, the matrix

$$B = A + \epsilon x y^T, \quad \epsilon > 0 \tag{37}$$

possesses the Perron-Frobenius property and for the spectral radii there holds

$$\rho(A) < \rho(B). \tag{38}$$

Moreover, if A possesses the strong Perron-Frobenius property, then so does B.

Proof: Let $\rho(A) = \lambda_1 \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ be the eigenvalues of A. Consider first that: *The Jordan canonical form of A is diagonal.* Let $x^{(i)}$, $i = 1, 2, \cdots, n$, be the linearly independent eigenvectors of A corresponding to λ_i where $x^{(1)} = x$. Then,

$$Bx = (A + \epsilon xy^T)x = \left(\rho(A) + \epsilon y^T x\right) x.$$

Thus, $\rho(A) + \epsilon y^T x$ eigenvalue of B, and $\rho(A) < \rho(B)$. Let

$$\tilde{x}^{(i)} = x - \alpha_i x^{(i)}, \quad \alpha_i = \frac{\lambda_1 - \lambda_i + \epsilon y^T x}{\epsilon y^T x^{(i)}}, \; y^T x^{(i)} \neq 0.$$

Then,

$$B\tilde{x}_i = (A + \epsilon xy^T)(x - \alpha_i x^{(i)}) = (\lambda_1 + \epsilon y^T x - \alpha_i \epsilon y^T x^{(i)})x - \lambda_i \alpha_i x^{(i)} = \lambda_i (x - \alpha_i x^{(i)}$$

Thus λ_i eigenvalue of B. If $y^T x^{(i)} = 0$ we chose $\tilde{x}^{(i)} = x^{(i)}$, while if $\lambda_i = \lambda_1$ we chose $\alpha_i = \frac{y^T x}{y^T x^{(i)}}$, to get the same result.

Remark (2)

Based on continuity properties we can conclude that the last result is valid also for a cone of directions around xy^T.

M matrices

Definition (15)

A matrix $A \in \mathbf{R}^{n,n}$ is said to be a Z matrix if its diagonal entries are nonnegative and its off diagonal entries are nonpositive.

Definition (16)

A matrix $A \in \mathbf{R}^{n,n}$ is said to be an M matrix if it can be written in the form

$$A = sI - B, \quad B \geq 0, \quad s \geq \rho(B).$$

Obviously, an M matrix is a Z matrix.

Definition (17)

A matrix $T \in \mathbf{R}^{n,n}$ is said to be convergent if

$$\rho(T) < 1.$$

Theorem (34)

A matrix $T \geq 0$ is convergent **iff** $(I - T)^{-1}$ *exists and*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k \geq 0. \tag{39}$$

Proof: Let T be convergent, then from the identity

$$(I - T)(I + T + T^2 + \cdots + T^k) = I - T^{k+1}$$

we obtain (39), since $\lim_{k\to\infty} T^{k+1} = 0$.
Conversely, if $(I - T)^{-1}$ exists and (39) holds, let $x \geq 0$ be such that $Tx = \rho(T)x$. Then $\rho(T) \neq 1$ since $(I - T)^{-1}$ exists and

$$(I - T)x = (1 - \rho(T))x \Rightarrow (I - T)^{-1}x = \frac{1}{1 - \rho(T)}x.$$

Then since $x \geq 0$ and $(I - T)^{-1} \geq 0$, it follows that $\rho(T) < 1$.

Theorem (35)

A matrix $A \in R^{n,n}$ is a nonsingular M matrix **iff** *A is a Z matrix and $A^{-1} \geq 0$.*

Proof: Let A be a nonsingular M matrix, then A is written as

$$A = sI - B, \quad B \geq 0, \quad s > \rho(B).$$

It follows that

$$A^{-1} = (sI - B)^{-1} = \frac{1}{s}\left(I - \frac{1}{s}B\right)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{1}{s}B\right)^{k} \geq 0.$$

Conversely, if A is a Z matrix and $A^{-1} \geq 0$, obviously A is nonsingular and is written as $A = sI - B,\ B \geq 0,\ s \geq 0$. Then, from $A^{-1} = \frac{1}{s}\left(I - \frac{1}{s}B\right)^{-1} \geq 0$ and from Theorem (34) it follows that $\frac{1}{s}B$ is convergent, which means that $\rho\left(\frac{B}{s}\right) < 1 \Leftrightarrow s > \rho(B)$ and thus A is a nonsingular M matrix.

Theorem (36)

A matrix $A \in \mathbb{R}^{n,n}$ is an irreducible and nonsingular M matrix **iff** *A is a Z matrix and $A^{-1} > 0$.*

Proof: Let A be a nonsingular M matrix, then A is written as

$$A = sI - B, \quad B \geq 0, \quad s > \rho(B),$$

where B is irreducible. Thus, $B^k > 0$ for large enough k. Then,

$$A^{-1} = (sI - B)^{-1} = \frac{1}{s}\left(I - \frac{1}{s}B\right)^{-1} = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{1}{s}B\right)^k > 0.$$

Conversely, if A is a Z matrix and $A^{-1} > 0$, obviously A is nonsingular and is written as $A = sI - B$, $B \geq 0$, $s \geq 0$. Then, from $A^{-1} = \frac{1}{s}\left(I - \frac{1}{s}B\right)^{-1} > 0$ and from Theorem (34) it follows that $\frac{1}{s}B$ is convergent, which means that $\rho\left(\frac{B}{s}\right) < 1 \Leftrightarrow s > \rho(B)$ and the powers B^k are positive for large enough k. Thus, A is an irreducible and nonsingular M matrix.

M matrices

Exercise (8)

Prove that any principal submatrix of an M matrix $A \in \mathbf{R}^{n,n}$ is also an M matrix.

Exercise (9)

Prove that the real parts of the eigenvalues of an M matrix $A \in \mathbf{R}^{n,n}$, are nonnegative.

Exercise (10)

Prove that the $n \times n$ Laplace matrix

$$A = \begin{bmatrix} 2 & -1 & \cdots & 0 \\ -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 2 \end{bmatrix},$$

is a nonsingular M matrix.

Exercise (11)

Prove that the $n \times n$ matrix

$$A = \begin{bmatrix} 2 & -2 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -2 & 2 \end{bmatrix},$$

is a singular M matrix.

Exercise (12)

Prove that the $n \times n$ matrix

$$A = \begin{bmatrix} 2 & -\frac{3}{2} & 0 & \cdots & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{3}{2} & \cdots & 0 & 0 \\ 0 & -\frac{3}{2} & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 2 & -\frac{3}{2} \\ 0 & 0 & 0 & \cdots & -\frac{3}{2} & 2 \end{bmatrix},$$

is not an M matrix.

Definition (18)

A matrix $A \in \boldsymbol{R}^{n,n}$ is said to be a Stieltjes matrix if it is Z, symmetric and positive definite matrix.

Theorem (37)

A Stieltjes matrix $A \in \boldsymbol{R}^{n,n}$ is a nonsingular M matrix.

Proof: Since A is positive definite, it is nonsingular. Since A is a Z matrix, it is written as $A = sI - B$, $B \geq 0$, $s \geq 0$. Let $x \geq 0$ be the normalized eigenvector of B corresponding to $\rho(B)$. Then

$$x^T Ax = sx^T x - x^T Bx = s - \rho(B) > 0 \Rightarrow s > \rho(B).$$

Thus, A is an M matrix.

Generalized M matrices (GM matrices)

Definition (19)

A matrix $A \in \mathbb{R}^{n,n}$ is said to be a GM matrix if it can be written as $A = sI - B, \; s \geq \rho(B)$, where B and B^T possess the Perron Frobenius property.

Theorem (38)

A matrix $A \in \mathbb{R}^{n,n}$ is a nonsingular GM matrix **iff** *A^{-1} and A^{-T} possess both the Perron Frobenius property and all the eigenvalues of A^{-1} have positive real parts.*

A. Elhashash, D. B. Szyld *Generalizations of M-matrices which may not have a nonnegative inverse* Linear Algebra Appl., In Press,

Applications of M and Stieltjes matrices

- Ordinary and partial differential equations
- Integral equations
- Economics
- Linear complementarity problems in operations research
- Markov chains
- Probability and Statistics

Splittings of matrices

For the solution of the nonsingular linear system

$$Ax = b, \quad A \in \boldsymbol{R}^{n,n}, \quad x, b \in \boldsymbol{R}^n, \tag{40}$$

consider the splitting

$$A = M - N, \tag{41}$$

where M is nonsingular. Then (40) becomes

$$Mx = Nx + b \Leftrightarrow x = M^{-1}Nx + M^{-1}b.$$

This suggests us to use the iterative method

$$Mx^{(k+1)} = Nx^{(k)} + b, \quad k = 0, 1, 2, \ldots, \quad x^{(0)} \in \boldsymbol{R}^n$$

or

$$x^{(k+1)} = M^{-1}Nx^{(k)} + M^{-1}b, \quad k = 0, 1, 2, \ldots, \quad x(0) \in \boldsymbol{R}^n$$

The above iterative scheme converges to the solution of (40) **iff**

$$\rho(M^{-1}N) < 1.$$

Splittings of matrices

A splitting $A = M - N$ is called:

- M**-splitting if** M **is an** M**-matrix and** $N \geq 0$**,**
- **Regular splitting if** $M^{-1} \geq 0$ **and** $N \geq 0$**,**
- **Weak regular of** 1**st type if** $M^{-1} \geq 0$ **and** $M^{-1}N \geq 0$**,**
- **Weak regular of** 2**nd type if** $M^{-1} \geq 0$ **and** $NM^{-1} \geq 0$**,**
- **Nonnegative of** 1**st type if** $M^{-1}N \geq 0$**,**
- **Nonnegative of** 2**nd type if** $NM^{-1} \geq 0$**,**
- ***Perron-Frobenius of** 1**st type if** $M^{-1}N$ **possesses the PF property,**
- ***Perron-Frobenius of** 2**nd type if** NM^{-1} **possesses the PF property,**

*D. Noutsos. *On Perron-Frobenius property of matrices having some negative entries.* 412 (2006), 132–153.

Theorem (39)

Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a nonnegative splitting of the first type. Then the following properties are equivalent:
(1) $\rho(M^{-1}N) < 1$
(2) $A^{-1}N \geq 0$
(3) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$
(4) $\rho(M^{-1}N) = \frac{\rho(A^{-1}M)-1}{\rho(A^{-1}M)}$
(5) $A^{-1}M \geq 0$
(6) $A^{-1}N \geq M^{-1}N.$

Proof:

- $1 \Rightarrow 2: \ A^{-1}N = (M-N)^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \geq 0$, since $M^{-1}N \geq 0$ and $\rho(M^{-1}N) < 1$.

Nonnegative splittings - Convergence Theorems

- $2 \Rightarrow 3:\ M^{-1}N = (A+N)^{-1}N = (I+A^{-1}N)^{-1}A^{-1}N \geq 0$. $A^{-1}N \geq 0 \Rightarrow \lambda = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$, λ eigenvalue of $M^{-1}N$. $M^{-1}N \geq 0 \Rightarrow \rho(M^{-1}N) = \frac{\mu}{1+\mu}$, μ nonnegative eigenvalue of $A^{-1}N$. $\frac{r}{1+r}$ increases for $r \geq 0$, thus $\mu = \rho(A^{-1}N)$.
- $3 \Rightarrow 4:\ A^{-1}N = A^{-1}(M-A) = A^{-1}M - I \Rightarrow \rho(A^{-1}N) = \lambda_{A^{-1}M} - 1$. Substituting it to property (3), we get $\rho(M^{-1}N) = \frac{\lambda_{A^{-1}M}-1}{\lambda_{A^{-1}M}}$. (3) implies that $\rho(M^{-1}N) < 1$, and thus $\lambda_{A^{-1}M} \geq 1$. $\frac{r-1}{r}$ increases for $r \geq 1$, thus $\lambda_{A^{-1}M} = \rho(A^{-1}M)$.
- $4 \Rightarrow 5:\ A^{-1}M = (M-N)^{-1}M = (I - M^{-1}N)^{-1}$. From property (4) we have $\rho(M^{-1}N) < 1$, thus from Theorem (34) we obtain that $(I - M^{-1}N)^{-1} \geq 0$.
- $5 \Rightarrow 6:\ A^{-1}N = A^{-1}M - I = (I - M^{-1}N)^{-1} - I = M^{-1}N + (M^{-1}N)^2 + \cdots \geq M^{-1}N$.
- $6 \Rightarrow 1:\ A^{-1}N = (I - M^{-1}N)^{-1}M^{-1}N \geq M^{-1}N$. Thus, $\rho(A^{-1}N) = \frac{\rho(M^{-1}N)}{1-\rho(M^{-1}N)} \geq \rho(M^{-1}N) \Rightarrow \rho(M^{-1}N) < 1$.

Theorem (40)

Let $A \in \mathbf{R}^{n,n}$ be a nonsingular matrix with $A^{-1} \geq 0$ and $A = M_1 - N_1 = M_2 - N_2$ be two convergent nonnegative splittings of the first or of the second type, and $x \geq 0$, $y \geq 0$ the Perron-Frobenius eigenvectors, respectively such that

$$N_2 x \geq N_1 x \quad \text{or} \quad N_2 y \geq N_1 y \quad \text{with} \quad y > 0$$

then

$$\rho(M_1^{-1} N_1) \leq \rho(M_2^{-1} N_2). \tag{42}$$

Moreover, if $A^{-1} > 0$ and $N_2 \neq N_1$, then

$$\rho(M_1^{-1} N_1) < \rho(M_2^{-1} N_2). \tag{43}$$

Proof: Let $T_1 = M_1^{-1}N_1$ and $T_2 = M_2^{-1}N_2$. Assume $N_2x \geq N_1x$, Then

$$\begin{aligned} A^{-1}N_2x - A^{-1}N_1x &= (I - T_2)^{-1}T_2x - (I - T_1)^{-1}T_1x \\ &= (I - T_2)^{-1}T_2x - \frac{\rho(T_1)}{1-\rho(T_1)}x \geq 0. \end{aligned}$$

From Theorem (22) we obtain that

$$\rho\left((I - T_2)^{-1}T_2\right) = \frac{\rho(T_2)}{1-\rho(T_2)} \geq \frac{\rho(T_1)}{1-\rho(T_1)}$$

The monotinicity of the function $f(z) = \frac{z}{1-z}$ implies that $\rho(T_1) \leq \rho(T_2)$. The strict inequality is obvious.
The proof, when we assume $N_2y \geq N_1y, \ y > 0$, is analogous, using Theorem (23).

Theorem (41)

Let $A \in \mathbf{R}^{n,n}$ be a nonsingular matrix and $A = M_1 - N_1 = M_2 - N_2$ be two convergent nonnegative splittings of the first or of the second type, $T_1 = M_1^{-1}N_1$, $T_2 = M_2^{-1}N_2$ and $x \geq 0$, $y \geq 0$ the Perron-Frobenius eigenvectors. If either $N_1x \geq 0$ or $N_2y \geq 0$ with $y > 0$ and if

$$M_1^{-1} \geq M_2^{-1} \quad \textit{then}$$

$$\rho(T_1) \leq \rho(T_2). \tag{44}$$

Moreover, if $M_1^{-1} > M_2^{-1}$ and $N_2 \neq N_1$, then $\rho(T_1) < \rho(T_2)$.

Proof: $M_1^{-1}N_1x = T_1x = \rho(T_1)x \Leftrightarrow M_1x = \frac{1}{\rho(T_1)}N_1x \geq 0$.
$Ax = M_1(I - T_1)x = (1 - \rho(T_1))M_1x = \frac{1-\rho(T_1)}{\rho(T_1)}N_1x \geq 0$.
$(M_1^{-1} - M_2^{-1})Ax = (I - T_1)x - (I - T_2)x = T_2x - \rho(T_1)x \geq 0$.
From Theorem (22) we obtain $\rho(T_1) \leq \rho(T_2)$.

Theorem (42)

Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ be a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. Then the following properties are equivalent:
(1) $\rho(M^{-1}N) < 1$
(2) $A^{-1}N$ *possesses the Perron-Frobenius property*
(3) $\rho(M^{-1}N) = \frac{\rho(A^{-1}N)}{1+\rho(A^{-1}N)}$
(4) $\rho(M^{-1}N) = \frac{\rho(A^{-1}M)-1}{\rho(A^{-1}M)}$
(5) $A^{-1}Mx \geq x$
(6) $A^{-1}Nx \geq M^{-1}Nx$.

Theorem (43)

Let $A \in \mathbb{R}^{n,n}$ be a nonsigular matrix and the splitting $A = M - N$ is a Perron-Frobenius splitting, with x the Perron-Frobenius eigenvector. If one of the following properties holds true:
(i) There exists $y \in \mathbb{R}^n$ such that $A^T y \geq 0$, $N^T y \geq 0$ and $y^T A x > 0$
(ii) There exists $y \in \mathbb{R}^n$ such that $A^T y \geq 0$, $M^T y \geq 0$ and $y^T A x > 0$
then

$$\rho(M^{-1}N) < 1.$$

Theorem (44)

Let $A \in \mathbb{R}^{n,n}$ be a nonsingular matrix and $A = M_1 - N_1$, $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of the first and second type, respectively and $x \geq 0$, $y \geq 0$ the Perron-Frobenius eigenvectors, such that

$$y^T A^{-1} \geq 0, \quad y^T x > 0 \quad \text{and} \quad N_2 x \geq N_1 x.$$

then

$$\rho(T_1) \leq \rho(T_2). \tag{45}$$

Moreover, if $y^T A^{-1} > 0$ and $N_2 x \neq N_1 x$, then

$$\rho(T_1) < \rho(T_2). \tag{46}$$

Theorem (45)

Let $A \in \boldsymbol{R}^{n,n}$ be a nonsingular matrix and $A = M_1 - N_1$, $A^T = M_2^T - N_2^T$ are two convergent Perron-Frobenius splittings of the first and second kind, respectively and $x \geq 0$, $y \geq 0$ the Perron-Frobenius eigenvectors, such that

$$N_1 x \geq 0 \;\; \text{and} \;\; y^T M_1^{-1} \geq y^T M_2^{-1}, \;\; y^T x > 0.$$

Then

$$\rho(T_1) \leq \rho(T_2). \tag{47}$$

Moreover, if $y^T M_1^{-1} > y^T M_2^{-1}$ and $N_1 x \neq 0$, the inequality is strict, while if $y^T M_1^{-1} = y^T M_2^{-1}$, it becomes equality.

Example (12)

We consider the splittings
$A = M_1 - N_1 = M_2 - N_2 = M_3 - N_3 = M_4 - N_4 = M_5 - N_5$ where

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

The above splittings are convergent ones with

$$\rho(T_2) = 0 < \rho(T_1) = \rho(T_3) = \rho(T_4) = \frac{1}{3} < \rho(T_5) = 0.4472.$$

Example

The first four splittings are Perron-Frobenius splittings while the last one is a nonnegative splitting. The same holds for $A^T = M_1^T - N_1^T = M_3^T - N_3^T = M_4^T - N_4^T = M_5^T - N_5^T$.

$1 \leftrightarrow 2$: Both Theorems hold:

$$\rho(T_2) \leq \rho(T_1).$$

$1 \leftrightarrow 3$: Both Theorems hold:

$$\rho(T_1) = \rho(T_3).$$

$3 \leftrightarrow 2$: The same with $1 \leftrightarrow 2$:

$$\rho(T_2) \leq \rho(T_3).$$

Perron-Frobenius splittings

Example

$3 \leftrightarrow 4$: The same with $1 \leftrightarrow 3$:

$$\rho(T_3) = \rho(T_4).$$

$4 \leftrightarrow 2$: The same with $1 \leftrightarrow 2$:

$$\rho(T_2) \leq \rho(T_4).$$

$4 \leftrightarrow 5$: Both Theorems hold with strict inequalities:

$$\rho(T_4) < \rho(T_5).$$

$5 \leftrightarrow 2$: Theorem (44) holds with strict inequality:

$$\rho(T_2) < \rho(T_5).$$

The Stein-Rosenberg Theorem

Theorem (46)

Let the Jacobi matrix $B \equiv L + U$ be a nonnegative $n \times n$ matrix with zero diagonal entries, where L and U are the lower and upper triangular parts of B, respectively, and let L_1 be the Gauss-Seidel matrix. Then one and only one of the following mutually exclusive relations is valid:

(i) $\rho(B) = \rho(L_1) = 0$.

(ii) $0 < \rho(L_1) < \rho(B) < 1$.

(iii) $\rho(B) = \rho(L_1) = 1$.

(iv) $1 < \rho(B) < \rho(L_1)$.

P. Stein and R.L.Rosenberg,
On the solution of linear simultaneous equations by iteration. J. London Math. Soc. 23 (1948), 111–118.

The Stein-Rosenberg theorem on nonnegative splittings

Theorem (47)

Let $A \in \mathbb{R}^{n,n}$ and the splittings $A = M_1 - N_1 = M_2 - N_2$ be both nonnegative splittings, $(M_i^{-1}N_i \geq 0, \quad i = 1, 2)$ and

$$M_1^{-1}N_1 \geq M_1^{-1}N_2 \geq 0, \quad N_1 \neq N_2, \quad N_2 \neq 0. \tag{48}$$

Assume that the matrices $M_1^{-1}N_1$, $T = M_1^{-1}(N_1 - N_2)$ and $F = M_1^{-1}N_2$ are up to a permutation, using the same permutation matrix, of the form

$$M_1^{-1}N_1 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & 0 \end{pmatrix} \tag{49}$$

with P_{11}, T_{11} and F_{11} being $k \times k$ matrices $(k \leq n)$, P_{11} irreducible and $T_{11}, F_{11} \neq 0$. Then exactly one of the following statements holds:
(i) $0 < \rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1$
(ii) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$
(iii) $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1.$
If $T_{11} = 0$ the second inequality of (i) and the first one of (iii) become equalities, while if $F_{11} = 0$ the first inequality of (i) becomes equality.

The Stein-Rosenberg thm on Perron-Frobenius splittings

Theorem (48)

Let $A \in \boldsymbol{R}^{n,n}$ and the splittings $A = M_1 - N_1 = M_2 - N_2$ be Perron-Frobenius splittings of the second and first kind, respectively, with y_1, x_2 being the associated left and right Perron eigenvectors and

$$M_1^{-1}N_1x_2 \geq M_1^{-1}N_2x_2 \geq 0, \quad M_1^{-1}N_1x_2 \neq M_1^{-1}N_2x_2 \neq 0. \quad (50)$$

Assume that the matrices $M_1^{-1}N_1, \quad T = M_1^{-1}(N_1 - N_2)$ and $F = M_1^{-1}N_2$ are up to a permutation, using the same permutation matrix, of the form

$$M_1^{-1}N_1 = \begin{pmatrix} P_{11} & 0 \\ P_{21} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & 0 \\ T_{21} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & 0 \\ F_{21} & 0 \end{pmatrix}$$

with $P_{11}, \; T_{11}$ and F_{11} being $k \times k$ matrices $(k \leq n)$, P_{11}^T possesses the strong Perron-Frobenius property and $T_{11}, F_{11} \neq 0$.

The Stein-Rosenberg thm on Perron-Frobenius splittings

Theorem (continuous of 48)

Then exactly one of the following statements holds:
(i) $0 < \rho(M_2^{-1}N_2) < \rho(M_1^{-1}N_1) < 1$
(ii) $\rho(M_2^{-1}N_2) = \rho(M_1^{-1}N_1) = 1$
(iii) $\rho(M_2^{-1}N_2) > \rho(M_1^{-1}N_1) > 1$.
If $T_{11} = 0$ *the second inequality of (i) and the first one of (iii) become equalities, while if* $F_{11} = 0$ *the first inequality of (i) becomes equality.*

D. Noutsos
On Stein-Rosenberg type theorems for nonnegative and Perron-Frobenius splittings. Linear Algebra Appl. (2008), In press.

Linear Autonomous Differential System:

$$\frac{dx}{dt} = Ax(t), \quad A \in \mathbb{R}^{n \times n}, \quad x(0) = x_0 \in \mathbb{R}^n, \quad t \geq 0,$$

whose solution (**trajectory**) is given by

$$x(t) = e^{tA} x_0 \quad t \geq 0.$$

Characterize:

• Eventually exponentially nonnegative matrices $(e^{tA} \geq 0 \;\; \forall t \geq t_0)$.

• Trajectories that become nonnegative at a finite time (*reachability of* $\mathbb{R}^n_+$)

• Trajectories that remain nonnegative for all time thereafter (*holdability of* $\mathbb{R}^n_+$).

Exponentially nonnegative matrices

Definition (19)

An $n \times n$ matrix $A = [a_{ij}]$ is called:

- essentially nonnegative *(*positive*), if $a_{ij} \geq 0$ $(a_{ij} > 0)$ for all $i \neq j$;*
- exponentially nonnegative *(*positive*) if $\forall t \geq 0$,*
  $$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \geq 0 \ \ (e^{tA} > 0);$$
- eventually exponentially nonnegative *(*positive*) if $\exists t_0 \in [0, \infty)$ such that $\forall t \geq t_0$, $e^{tA} \geq 0$ $(e^{tA} > 0)$. We denote the smallest such nonnegative number by $t_0 = t_0(A)$ and refer to it as the* exponential index *of A.*

Theorem (49)

$A \in \mathbb{R}^{n\times n}$ is exponentially nonnegative if and only if A is essentially nonnegative.

If A is essentially nonnegative, then there exists $\alpha \geq 0$ such that $A + \alpha I \geq 0$. Hence, as A and αI commute, we have that for all $t \geq 0$,

$$e^{tA} = e^{-t\alpha I} e^{t(A+\alpha I)} = e^{-t\alpha} e^{t(A+\alpha I)} \geq 0.$$

Conversely, let $e^{tA} \geq 0$ for all $t \geq 0$. Looking for contradiction suppose that $a_{ij} < 0$ for some $i \neq j$. Then, denoting the entries of A^k by $a_{ij}^{(k)}$, we have

$$(e^{tA})_{ij} = ta_{ij} + \frac{t^2}{2!} a_{ij}^{(2)} + \frac{t^3}{3!} a_{ij}^{(3)} + \dots$$

Thus, letting $t \to 0^+$ we have that for some $t > 0$, $(e^{tA})_{ij} < 0$, a contradiction.

Theorem (50)

For a matrix $A \in \mathbb{R}^{n\times n}$ the following properties are equivalent:
(i) *There exists $\alpha \geq 0$ such that both matrices $A+\alpha I$ and $A^T+\alpha I$ have the strong Perron-Frobenius property.*
(ii) *$A+\alpha I$ is eventually positive for some $\alpha \geq 0$.*
(iii) *$A^T+\alpha I$ is eventually positive for some $\alpha \geq 0$.*
(iv) *A is eventually exponentially positive.*
(v) *A^T is eventually exponentially positive.*

D. Noutsos and M. Tsatsomeros
Reachability and holdability of nonnegative states. SIAM journal on Matrix Analysis and Applications, (2008) In press.

Proof: The equivalence of (i), (ii) and (iii) is given by Theorem (25) applied to $A + \alpha I$. We will prove the equivalence of (ii) and (iv), with the equivalence of (iii) and (v) being analogous:
Let $A + \alpha I$ be eventually positive and let k_0 be a positive integer such that $(A + \alpha I)^k > 0$ for all $k \geq k_0$. Then there exists large enough $t_0 > 0$ so that the first k_0 terms of the series

$$e^{t(A+\alpha I)} = \sum_{m=0}^{\infty} \frac{t^m (A + \alpha I)^m}{m!}$$

are dominated by the remaining terms, rendering every entry of $e^{t(A+\alpha I)}$ positive for all $t \geq t_0$.
It follows that $e^{tA} = e^{-ta} e^{t(A+\alpha I)} > 0$ for all $t \geq t_0$. That is, A is eventually exponentially positive.

Conversely, suppose A is eventually exponentially positive. As $(e^A)^k = e^{kA}$, it follows that e^A is eventually positive. Thus, by Theorem (25), e^A has the strong Perron-Frobenius property. Recall that

$$\sigma(e^A) = \{e^\lambda : \lambda \in \sigma(A)\}$$

and so $\rho(e^A) = e^\lambda$ for some $\lambda \in \sigma(A)$. Then for each $\mu \in \sigma(A)$ with $\mu \neq \lambda$, we have

$$e^\lambda > |e^\mu| = |e^{\mathrm{Re}\,\mu + i\mathrm{Im}\,\mu}| = |e^{\mathrm{Re}\,\mu}| \cdot |e^{i\mathrm{Im}\,\mu}| = e^{\mathrm{Re}\,\mu}.$$

Hence λ is the spectral abscissa of A, namely,

$$\lambda > \mathrm{Re}\,\mu \quad \forall\, \mu \in \sigma(A), \;\; \mu \neq \lambda.$$

This means that there exists large enough $\alpha > 0$ such that

$$\lambda + \alpha > |\mu + \alpha| \quad \forall \mu \in \sigma(A), \;\; \mu \neq \lambda.$$

As $A + \alpha I$ and e^A have the same eigenvectors, it follows that $A + \alpha I$ has the strong Perron-Frobenius property. By Theorem (25), we obtain that $A + \alpha I$ is eventually positive.

Example (13)

Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$, observe that

$$A^2 = \begin{bmatrix} 2 & 3 & 4 & 4 \\ 2 & 3 & 4 & 4 \\ 0 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 5 & 9 & 13 & 13 \\ 5 & 9 & 13 & 13 \\ 1 & 3 & 5 & 5 \\ 3 & 6 & 9 & 9 \end{bmatrix}.$$

A is an eventually positive matrix with $k_0 = 3$, thus A is an eventually exponentially positive matrix. It is confirmed by computing e^{tA} for $t = 1, 2$:

$$\begin{bmatrix} 5.04 & 6.36 & 8.68 & 8.68 \\ 4.04 & 7.36 & 8.68 & 8.68 \\ -0.47 & 2.79 & 5.04 & 4.04 \\ 2.79 & 3.57 & 6.36 & 7.36 \end{bmatrix}, \quad \begin{bmatrix} 71.27 & 134.14 & 198.02 & 198.02 \\ 70.27 & 135.14 & 198.02 & 198.02 \\ 18.50 & 45.38 & 71.27 & 70.27 \\ 45.38 & 88.76 & 134.14 & 135.14 \end{bmatrix}.$$

Eventually exponentially nonnegative matrices

Theorem (51)

For a matrix $A \in \mathbb{R}^{n \times n}$ the following properties are equivalent:
(i) *There exist $\alpha_1, \alpha_2 \geq 0$ such that both matrices $A + \alpha_1 I$ and $A + \alpha_2 I$ are eventually nonnegative.*
(ii) *There exist $\alpha_1, \alpha_2 \geq 0$ such that both matrices $A^T + \alpha_1 I$ and $A^T + \alpha_2 I$ are eventually nonnegative.*
(iii) *A is eventually exponentially nonnegative.*
(iv) *A^T is eventually exponentially nonnegative.*

Note that if there exist $\alpha_1, \alpha_2 \geq 0$ such that both matrices $A + \alpha_1 I$ and $A + \alpha_2 I$ are eventually nonnegative, then $A + \alpha I$ is eventually nonnegative for all $\alpha \geq \min\{\alpha_1, \alpha_2\}$.

Example (14)

$$A = \begin{bmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad A^2 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}.$$

$$A + I = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad (A+I)^3 = \begin{bmatrix} 4 & 4 & 9 & 1 \\ 4 & 4 & 17 & 11 \\ 0 & 0 & 14 & 13 \\ 0 & 0 & 13 & 14 \end{bmatrix}.$$

Thus, A is eventually exponentially nonnegative matrix. For illustration, we compute e^{tA} for $t = 1, 2$ to respectively be

$$\begin{bmatrix} 1.54 & 1.18 & 2.34 & -0.01 \\ 1.18 & 1.54 & 4.05 & 2.96 \\ 0 & 0 & 4.19 & 3.19 \\ 0 & 0 & 3.19 & 4.19 \end{bmatrix}, \quad \begin{bmatrix} 3.76 & 3.63 & 18.15 & 10.90 \\ 3.63 & 3.76 & 35.44 & 29.92 \\ 0 & 0 & 27.80 & 26.80 \\ 0 & 0 & 26.80 & 27.80 \end{bmatrix}.$$

Example (15)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}, \quad A^k = \begin{bmatrix} 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 2^{k-1} & 2^{k-1} & k2^{k-1} & k2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 2^{k-1} & 2^{k-1} \end{bmatrix}.$$

$$A + I = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ -1 & 1 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix}, \quad (A+I)^2 = \begin{bmatrix} 5 & 4 & 6 & 6 \\ 4 & 5 & 6 & 6 \\ -2 & 2 & 5 & 4 \\ 2 & -2 & 4 & 5 \end{bmatrix}.$$

A is eventually nonnegative but $A + \alpha I$ is not, $\forall \alpha > 0$. Thus, A is not an eventually exponentially nonnegative matrix.

$$e^A = \begin{bmatrix} 4.19 & 3.19 & 7.39 & 7.39 \\ 3.19 & 4.19 & 7.39 & 7.39 \\ -1 & 1 & 4.19 & 3.19 \\ 1 & -1 & 3.19 & 4.19 \end{bmatrix}, e^{3A} \begin{bmatrix} 202.2 & 201.2 & 1210.3 & 1210.3 \\ 201.2 & 202.2 & 1210.3 & 1210.3 \\ -3 & 3 & 202.2 & 201.2 \\ 3 & -3 & 201.2 & 202.2 \end{bmatrix}.$$

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